workshop in particle physics

development of electromagnetic showers

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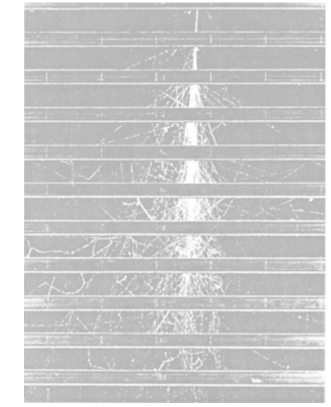
lectures outline

workshop in particle physics



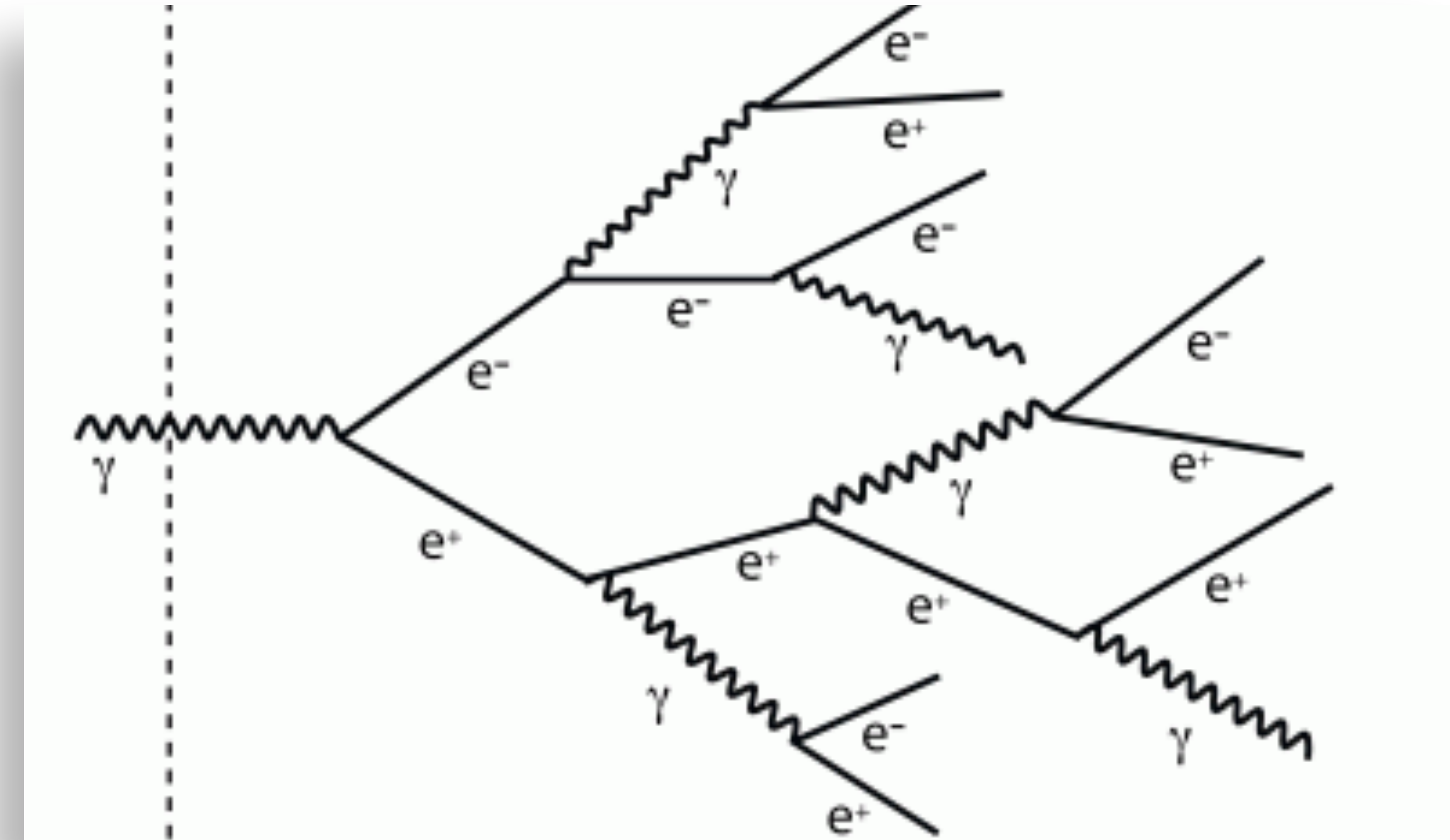
1. fundamental interactions & EM showers

2. **development of electromagnetic showers**

3. hadronic showers

outline
workshop in particle physics

- electron ionization losses
- electromagnetic showers
- cascade equations
- importance of energy losses
- universality of electromagnetic showers

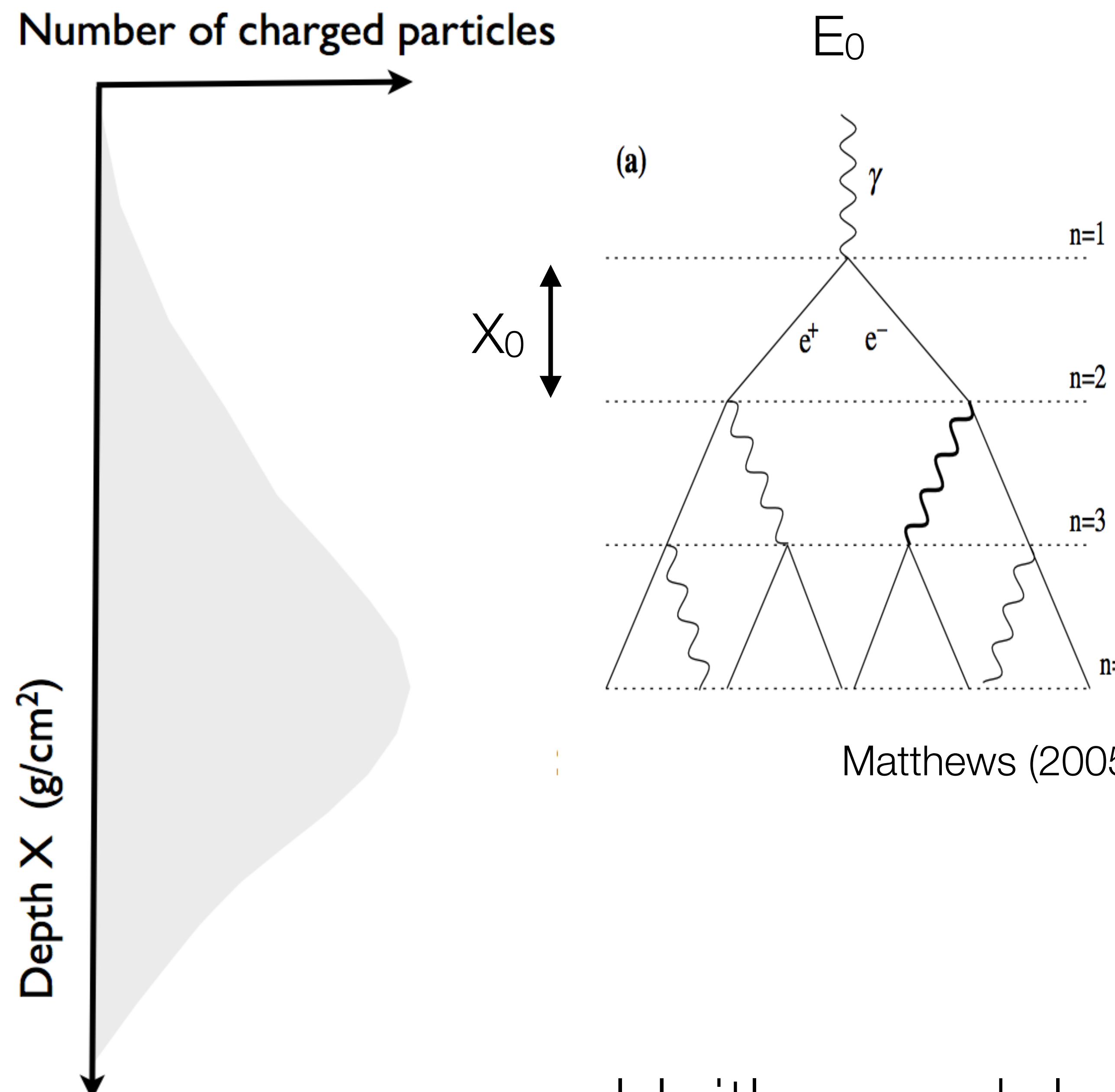


ELECTROMAGNETIC SHOWERS

electromagnetic showers

- particles involved
  - ***e*** $^{\pm}$ *and* ***γ***

- interactions involved
  - *bremsstrahlung*
  - *pair production*
  - *ionization losses*



Matthews (2005)

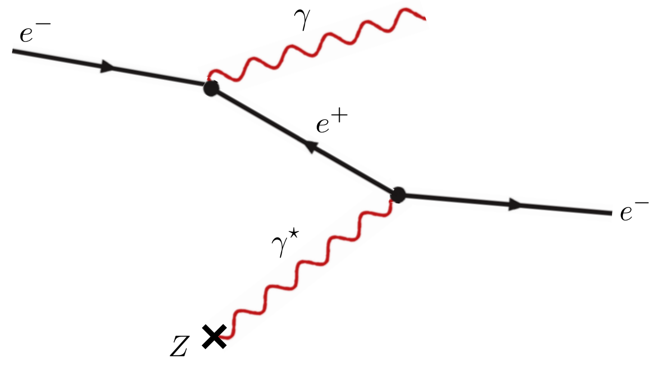
Heitler model

e-pair production & bremsstrahlung

electromagnetic showers

$$b \simeq \frac{1}{18 \ln 183 Z^{-1/3}} \qquad b_{air} \simeq 0.0135$$

BREMSSTRAHLUNG

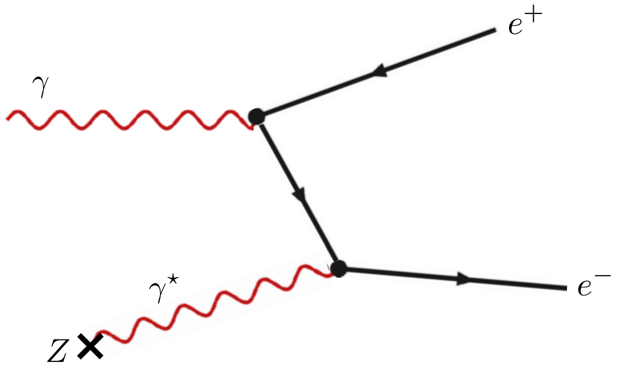


$$e + \gamma^\star \longrightarrow e + \gamma$$

$$\left.\frac{d\sigma}{dv}\right|_{e\to e+\gamma}(v, E_e) = 4Z^2\alpha r_e^2 \frac{1}{v}\left\{\left[1 + (1-v)^2 - \frac{2}{3}(1-v)\right]\frac{1}{18b} + \frac{1}{9}(1-v)\right\}$$

$$v = \frac{E_\gamma}{E_e}$$

PAIR PRODUCTION



$$\gamma + \gamma^\star \longrightarrow e^+ + e^-$$

$$\left.\frac{d\sigma}{du}\right|_{\gamma\to e^+ + e^-}(u, E_\gamma) = 4Z^2\alpha r_e^2 \left\{\left[u^2 + (1-u)^2 + \frac{2}{3}u(1-u)\right]\frac{1}{18b} - \frac{1}{9}u(1-u)\right\}$$

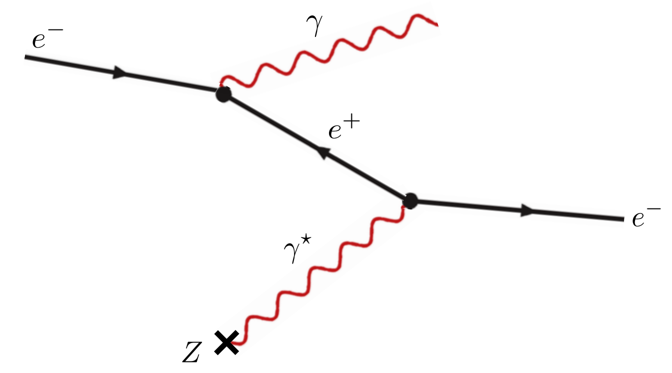
$$u = \frac{E_{e^+}}{E_\gamma}$$

high energy limit
full screening

e-pair production & bremsstrahlung

splitting function

BREMSSTRAHLUNG



$$e + \gamma^\star \longrightarrow e + \gamma$$

radiation length:
distance where the energy of an electron is reduced to E/e

$$\varphi(v) = \frac{d\sigma}{dv}(v)\Big|_{brems}\left(\frac{N_a}{A}\lambda_{rad}\right)$$

$$\varphi(v) = \frac{1}{v}\left[1 - \left(\frac{2}{3} - 2b\right)(1-v) + (1-v)^2\right]$$

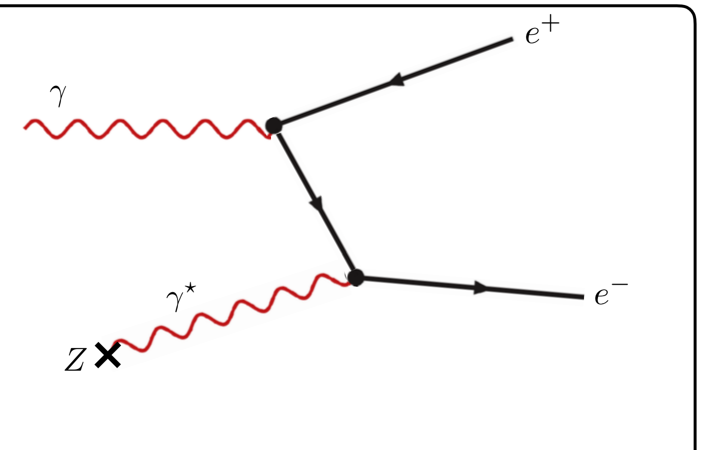
$$\int_0^1 dv\, v\, \varphi(v) = 1 + b$$

probability per unit of λ_{rad} that an $e^{\pm}$ of energy E_e emits a photon of energy $E_\gamma = vE_e$

e-pair production & bremsstrahlung

photon mean free path

PAIR PRODUCTION



photon mean free path:
9/7 of the radiation length

$$\psi(u) = \frac{d\sigma}{du}(u)\Big|_{epair}\left(\frac{N_A}{A}\lambda_{rad}\right)$$

$$\psi(u) = (1-u)^2 + \left(\frac{2}{3} - 2b\right)(1-u)u + u^2$$

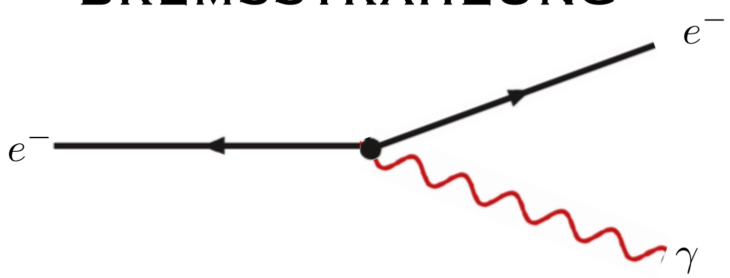
$$\sigma_0 = \int_0^1 du\,\psi(u) = \frac{7}{9} - \frac{b}{3}$$

probability per unit of λ_{rad} that a γ of energy E_γ produces a pair with e^+/e^- of energy $E_e = uE_\gamma$

electromagnetic showers

splitting functions

BREMSSTRAHLUNG



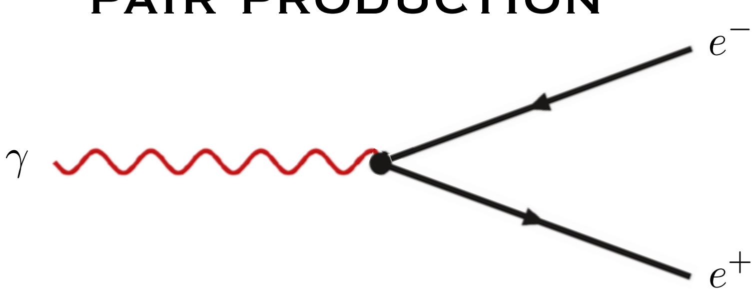
$$\varphi(v) = \frac{1}{v}\left[1 - \left(\frac{2}{3} - 2b\right)(1-v) + (1-v)^2\right]$$

$$\frac{1}{\lambda_{rad}} = \frac{N_A}{A} 4Z^2 \alpha r_e^2 \ln 183 Z^{-1/3}$$

$$\lambda_{rad}^{air} \simeq 37\,\mathrm{g/cm^2}$$

probability per unit of λ_{rad} that an $e^\pm$ of energy E_e emits a photon of energy $E_\gamma = vE_e$

PAIR PRODUCTION

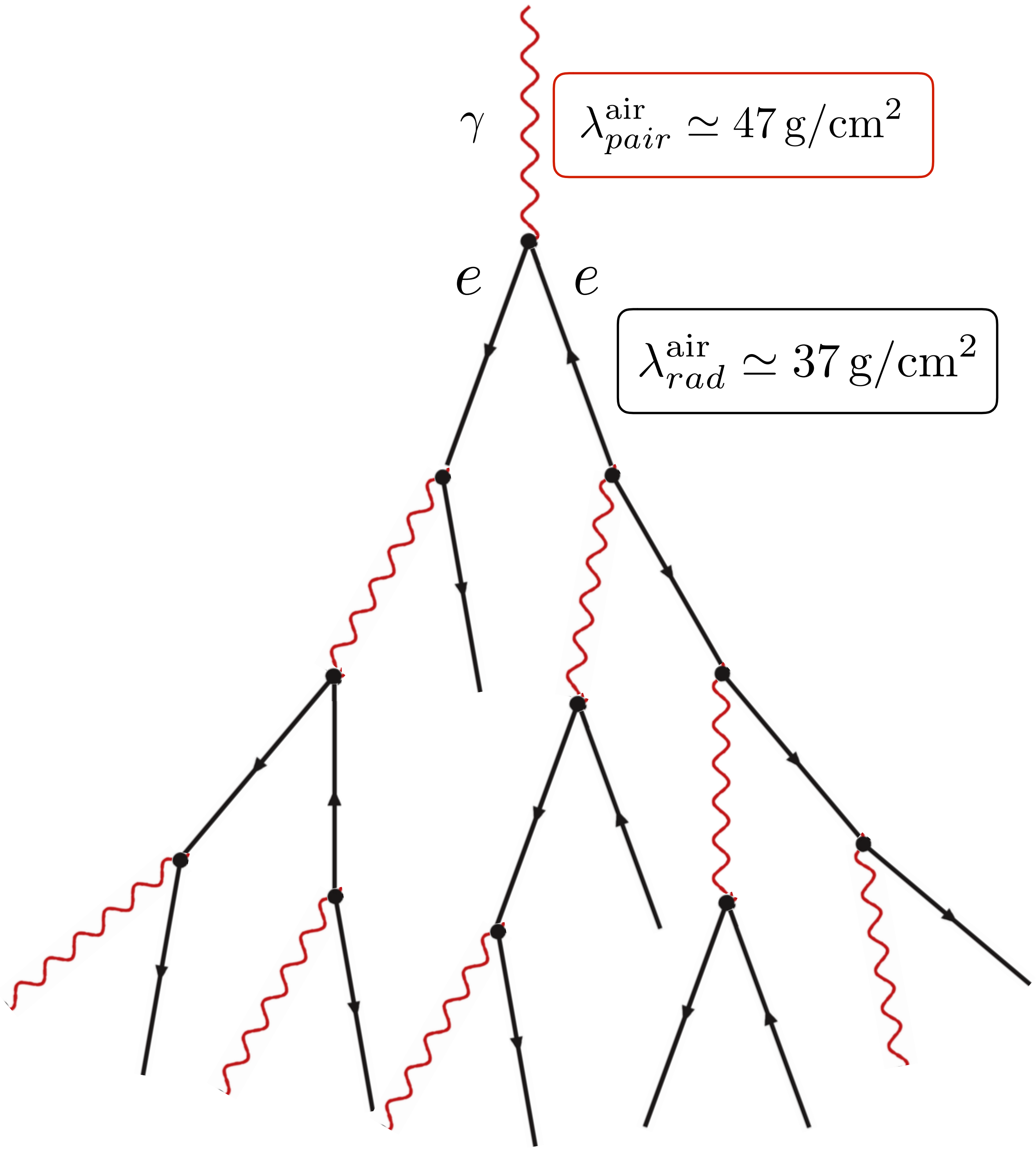


$$\psi(u) = (1-u)^2 + \left(\frac{2}{3} - 2b\right)(1-u)u + u^2$$

$$\lambda_{pair}^{air} \simeq 47\,\mathrm{g/cm^2}$$

probability per unit of λ_{rad} that a γ of energy E_γ produces a pair with e^+/e^- of energy $E_e = uE_\gamma$

electromagnetic showers



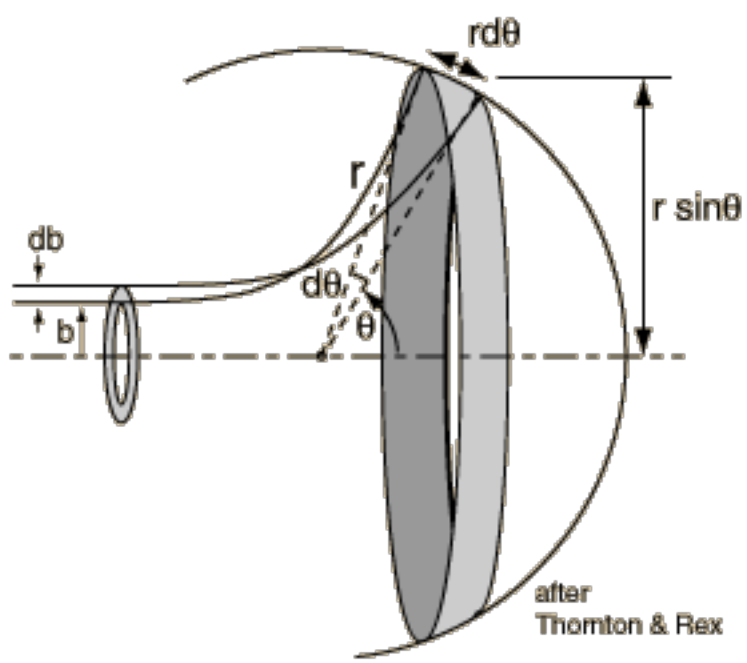
BREMSSTRAHLUNG $\varphi(v)$

PAIR PRODUCTION $\psi(u)$

electron lose energy by collision as well…

ionization losses

- electrons **collide** with atomic electrons and nuclei of the medium
- these collisions produce **energy loss** and **angular deviations**
- in first approximation the collisions can be described with **Rutherford scattering**

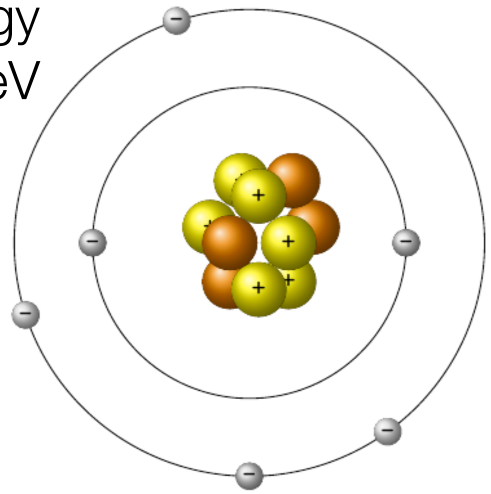


$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1\, Z_2\, e^2}{16\pi\, \epsilon_0\, E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{2E}\right)^2 \frac{1}{\sin^4 \frac{\theta}{2}}$$

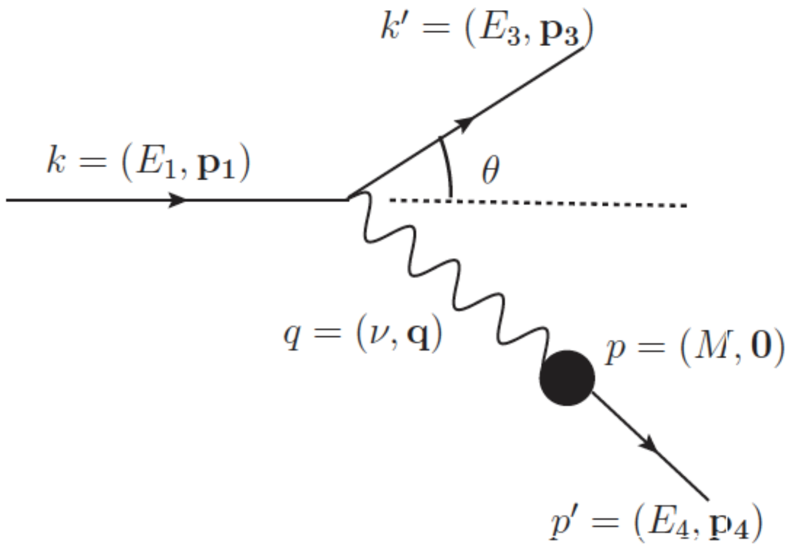
ionization losses

electron binding energy
~ 10-20 eV



- if energetic charged particle
  - **hit nuclei**: they change a bit direction (multiple scattering) & lose a bit of energy (bremsstrahlung)
  - **large hit**: inelastic collision (nuclei break up)
  - **hit electrons**: excite atoms or kick electron out (ionization)
  - **if electron get large kick**: delta rays

ionization losses



- a generalization of Rutherford scattering cross section

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{collision}} = \left(\frac{q_1 q_2}{Q^2}\right)^2 4m_e^2$$

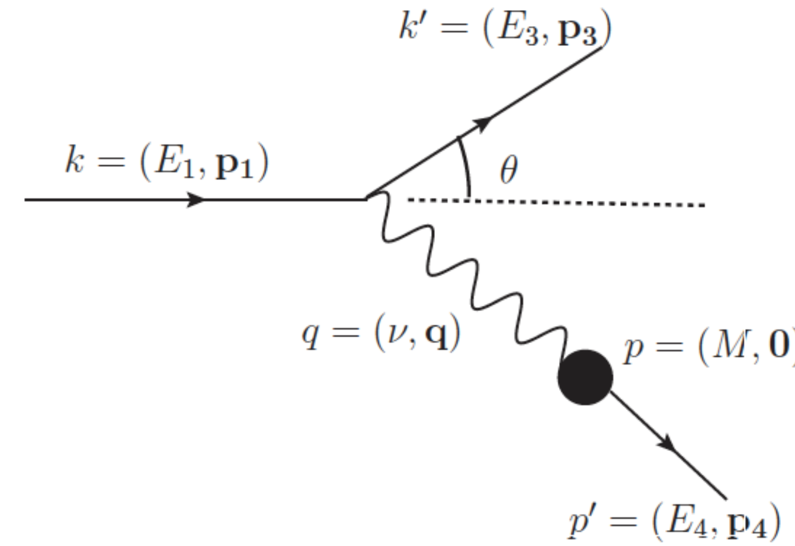
- with 4-momentum transferred $\quad Q^2 = -(k-k')^2 = -(p-p')^2$

$$-Q^2 = 2m_e - 2(E_1 E^2 - p_1 p_2 \cos\theta) = 2M^2 - 2ME'$$

$$-Q^2 \simeq 2p^2(1-\cos\theta) \simeq 2ME'$$

E' is the **kinetic energy transferred** to the target particle initially at rest

ionization losses



- a generalization of Rutherford scattering cross section

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{collision}} = \left(\frac{q_1 q_2}{Q^2}\right)^2 4m_e^2$$

$$\left.\frac{d\sigma}{dE}\right|_{\text{collision}} \propto \frac{1}{E^2}$$

$$\left.\frac{d\sigma}{dE}\right|_{\text{collision}} = 2\pi \frac{e^4}{m_e c^2 \beta^2 E^2}\left(1 - \beta^2 \frac{E}{E_{max}}\right)$$

ionization losses

- rate of energy loss per unit of column density

$$\left.\frac{dE}{dX}\right|_{\text{collision}} = \frac{ZN_A}{A}\int dE\,E\,\frac{d\sigma}{dE}$$

$$\left.\frac{dE}{dX}\right|_{\text{collision}} = \frac{ZN_A}{A}\,2\pi\frac{e^4}{m_ec^2}\frac{1}{\beta^2}\int_{E_{min}}^{E_{max}} dE\,E\,\frac{1}{E^2}\left[1-\beta^2\frac{E}{E_{max}}\right]$$

$$\left.\frac{dE}{dX}\right|_{\text{collision}} = \frac{ZN_A}{A}\,2\pi\frac{e^4}{m_ec^2}\frac{1}{\beta^2}\left[\ln\left(\frac{E_{max}}{E_{min}}\right)-\beta^2\right]$$

ionization losses

- rate of energy loss per unit of column density

$$\left.\frac{dE}{dX}\right|_{\rm collision} = \frac{ZN_A}{A}\,2\pi\frac{e^4}{m_e c^2}\,\frac{1}{\beta^2}\left[\ln\left(\frac{E_{max}}{E_{min}}\right) - \beta^2\right]$$

$$E_{min} \simeq \langle I \rangle$$

minimum energy transfer to an atomic electron (ionization)

$$E_{max} \simeq m_e c^2 \beta\gamma$$

kinematic upper limit for e-e scattering

ionization losses
Bethe-Bloch formula

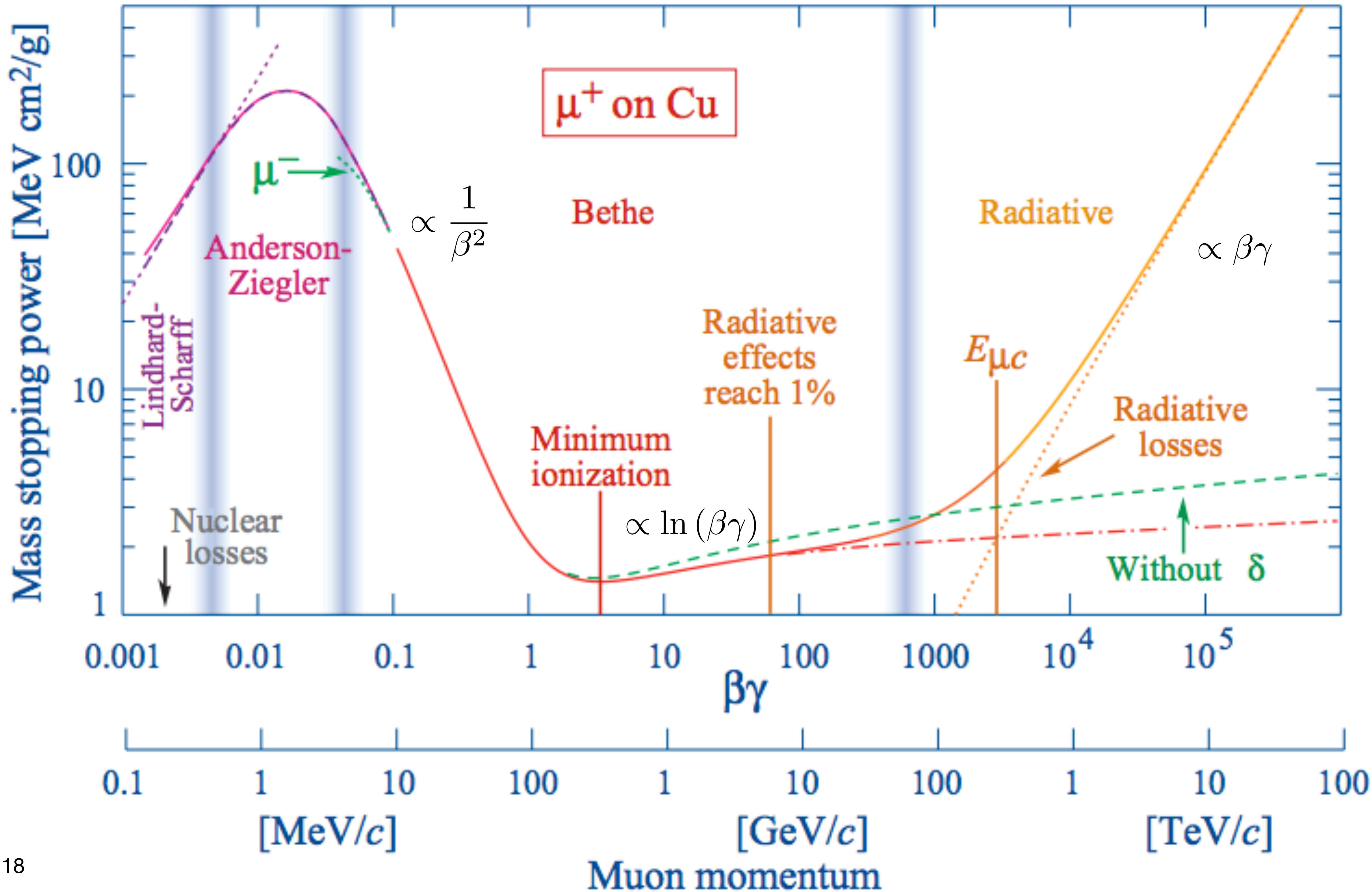
- rate of energy loss per unit of column density

$$\left.\frac{dE}{dX}\right|_{\text{collision}} = \frac{ZN_A}{A}\,2\pi\frac{e^4}{m_e c^2}\frac{1}{\beta^2}\left[\ln\left(\frac{E_{max}}{E_{min}}\right) - \beta^2\right]$$

$$\boxed{\left.\frac{dE}{dX}\right|_{\text{collision}} = \frac{ZN_A}{A}\,2\pi\frac{e^4}{m_e c^2}\frac{1}{\beta^2}\left[\ln\left(\frac{2m_e c^2\beta\gamma}{\langle I\rangle}\right) - \beta^2\right]}$$

$$\left.\frac{dE}{dX}\right|_{\text{collision}}^{\beta\gamma\ll 1} \propto \frac{1}{\beta^2} \qquad \left.\frac{dE}{dX}\right|_{\text{collision}}^{\beta\gamma\gg 1} \propto \ln(\beta\gamma) \approx \epsilon = \text{constant}$$

electron energy losses



ionization losses
Bethe-Bloch formula

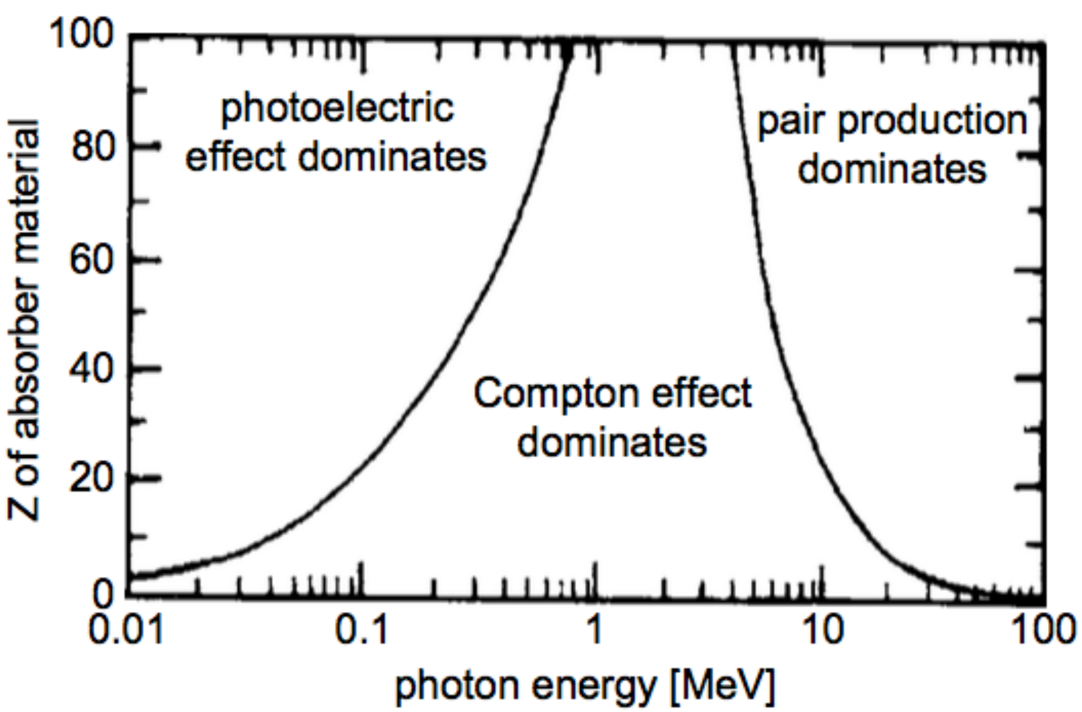
PDG Book

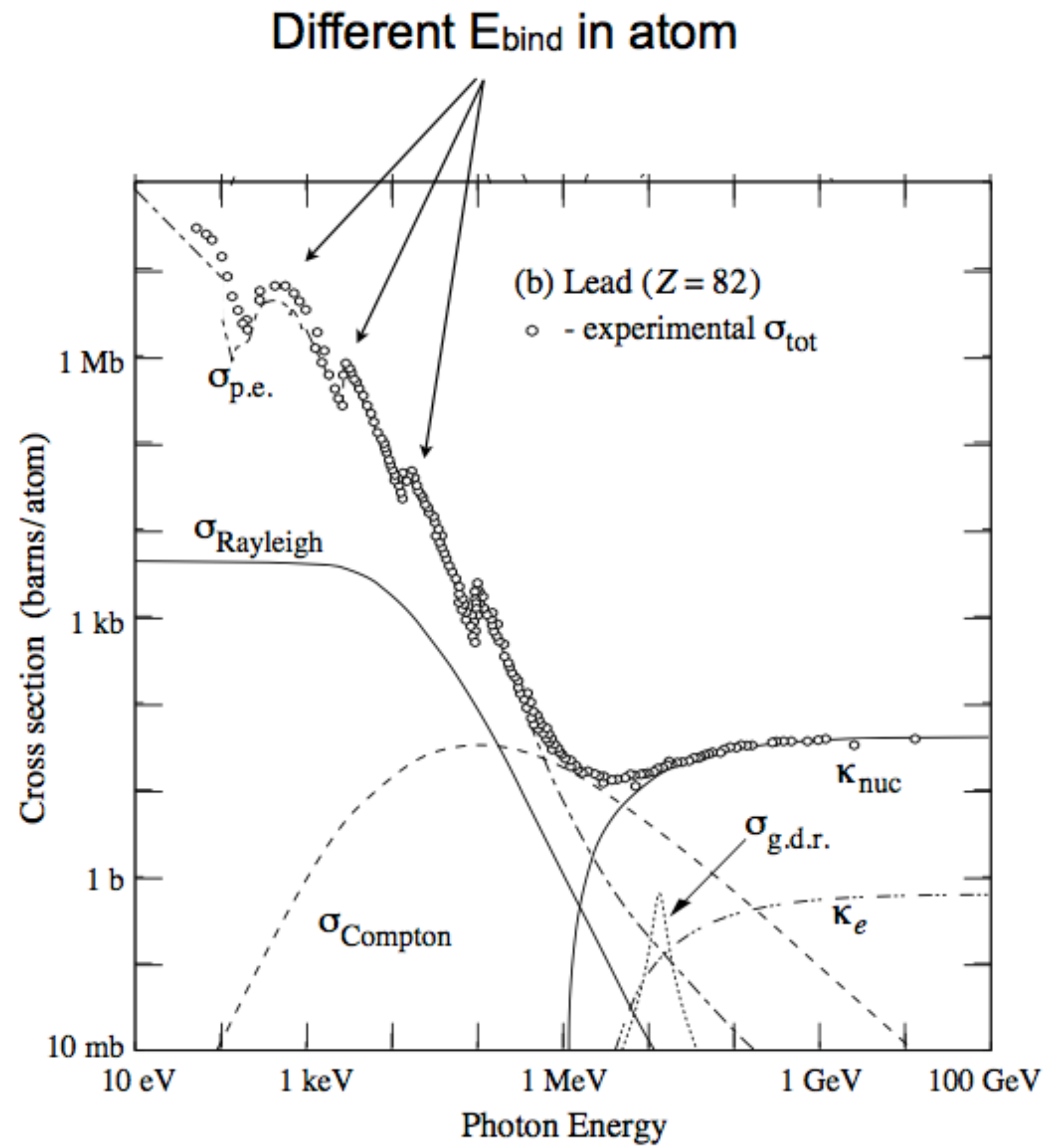
$$\left\langle -\frac{dE}{dx} \right\rangle = K z^2 \frac{Z}{A} \frac{1}{\beta^2} \left[\frac{1}{2} \ln \frac{2 m_e c^2 \beta^2 \gamma^2 W_{\max}}{I^2} - \beta^2 - \frac{\delta(\beta\gamma)}{2} \right]$$

$$W_{\max} = \frac{2 m_e c^2 \, \beta^2 \gamma^2}{1 + 2\gamma m_e / M + (m_e/M)^2}$$

maximum energy transfer in a collision

photon energy losses





energy losses for $e^{\pm}$

critical energy in air

$$\frac{dE}{dX} = \left.\frac{dE}{dX}\right|_{\mathrm{collision}} + \left.\frac{dE}{dX}\right|_{\mathrm{brems}}$$

$$\frac{dE}{dX} = \epsilon_{\mathrm{coll}} + \frac{E}{\lambda_{rad}}$$

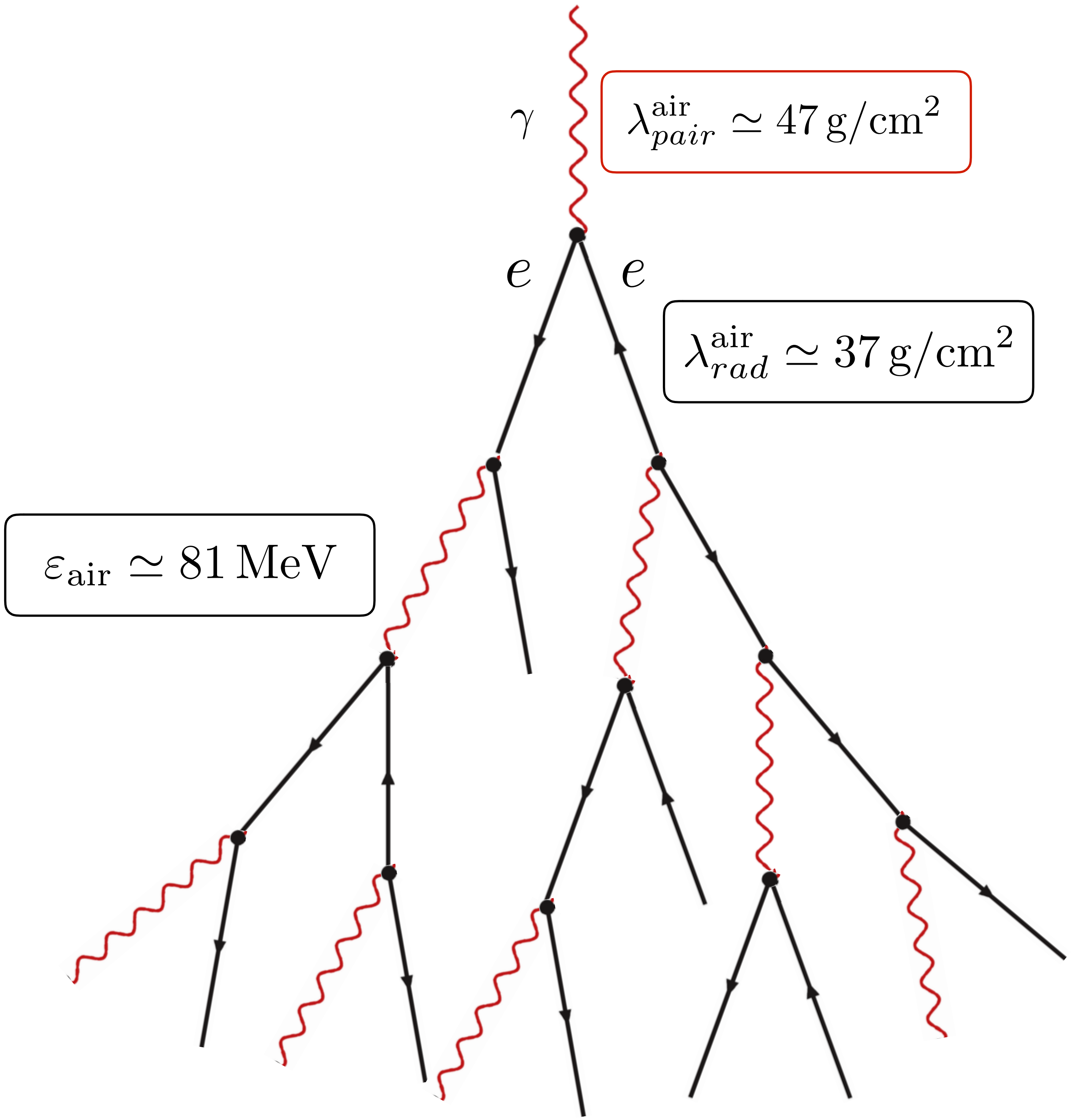
critical energy is when

$$\left.\frac{dE}{dX}\right|_{\mathrm{collision}} = \left.\frac{dE}{dX}\right|_{\mathrm{brems}}$$

$$\varepsilon_{\mathrm{air}} = \epsilon_{\mathrm{coll}} \times \lambda_{rad} \simeq 2.2 \frac{\mathrm{MeV}}{\mathrm{g\,cm^{-2}}} \times 37\mathrm{g\,cm^{-2}} \simeq 81\,\mathrm{MeV}$$

energy-independent loss per unit of radiation length

electromagnetic showers



BREMSSTRAHLUNG $\varphi(v)$

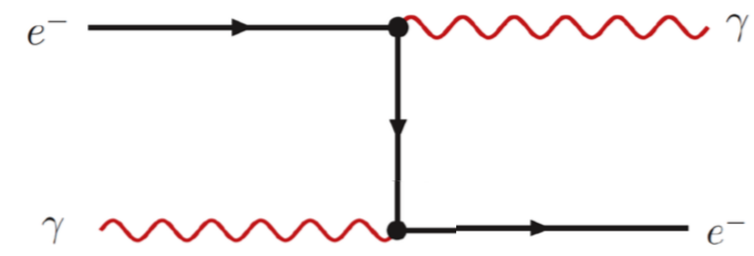
PAIR PRODUCTION $\psi(u)$

IONIZATION LOSSES $\left.\frac{dE}{dX}\right|_{\mathrm{collision}} = \frac{ZN_A}{A} 2\pi$

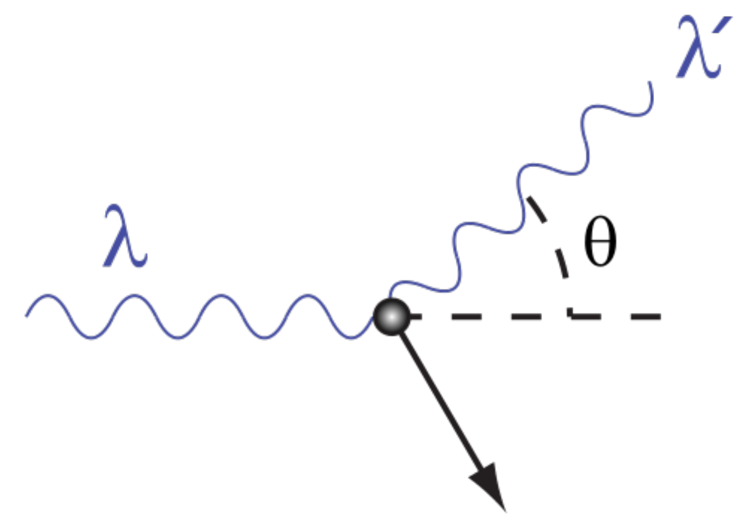
photons lose energy by Compton scattering as well…

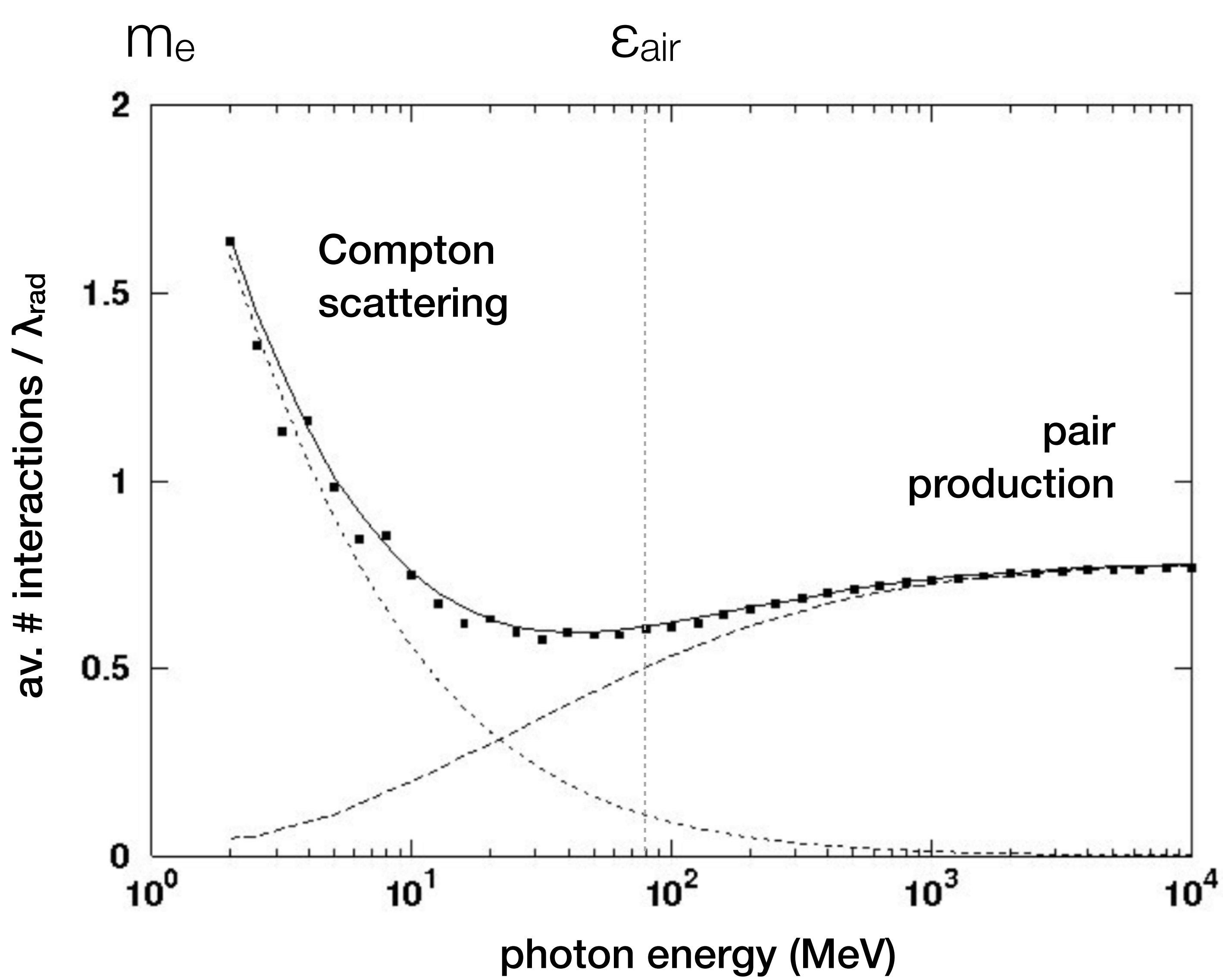
energy losses for γ

Compton scattering



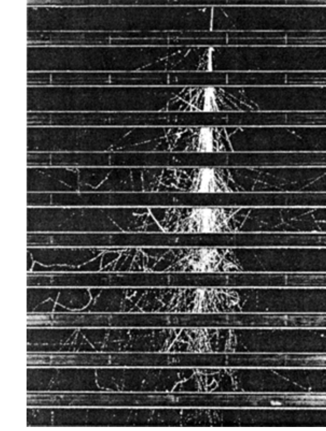
$$E_\gamma^{\text{final}} = \frac{E_\gamma^{\text{initial}}}{1 + \frac{E_\gamma^{\text{initial}}}{m_e c^2}(1 - \cos\theta)}$$





electromagnetic showers

cascade equations



OCTOBER, 1941 REVIEWS OF MODERN PHYSICS VOLUME 13

Cosmic-Ray Theory

BRUNO ROSSI AND KENNETH GREISEN

Cornell University, Ithaca, New York

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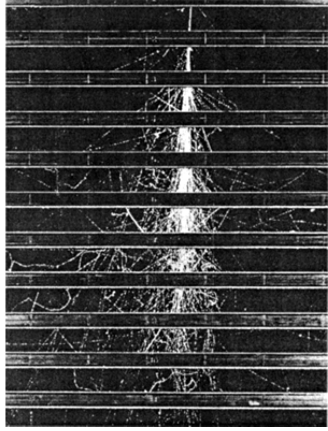
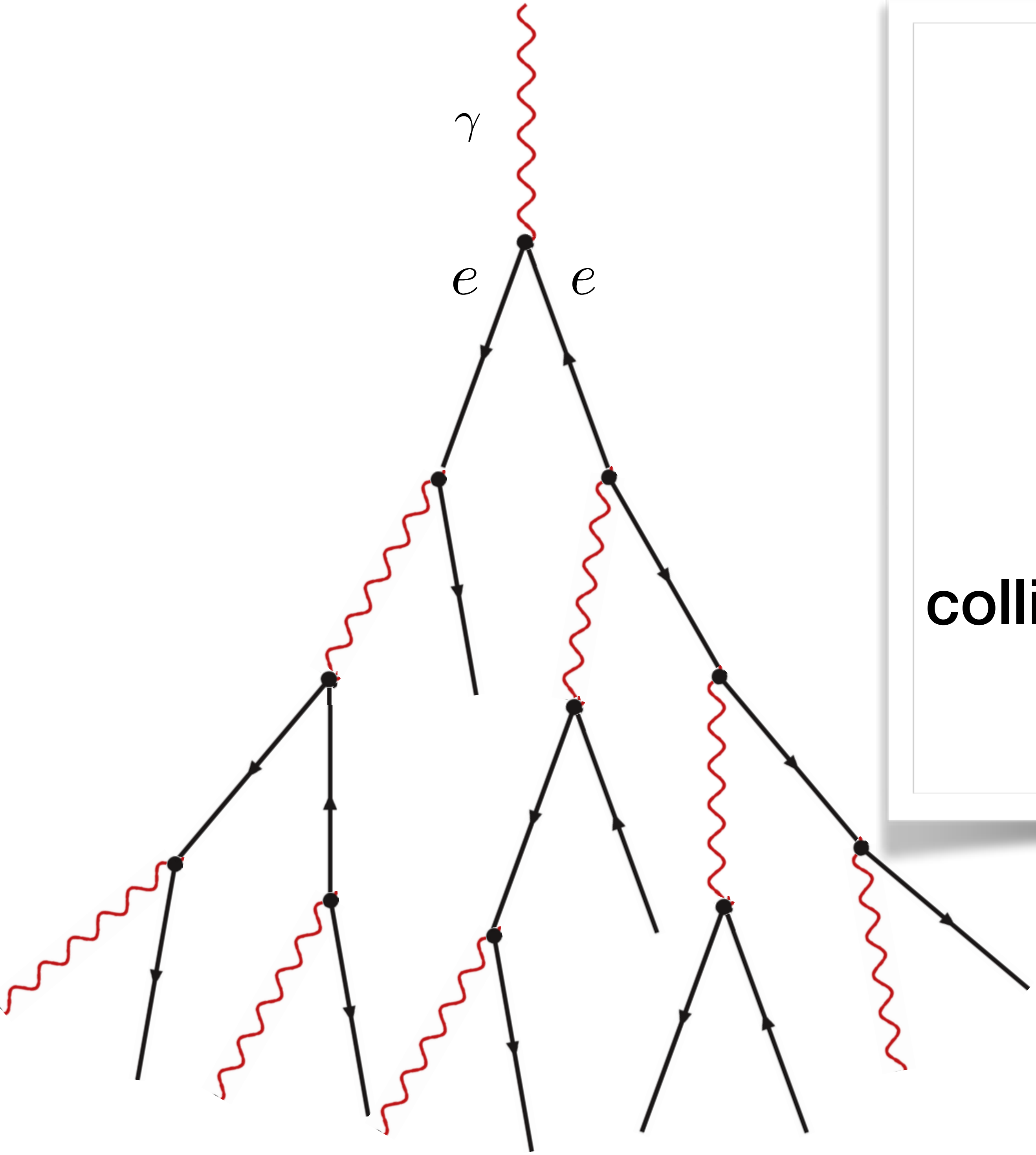
cascade equations



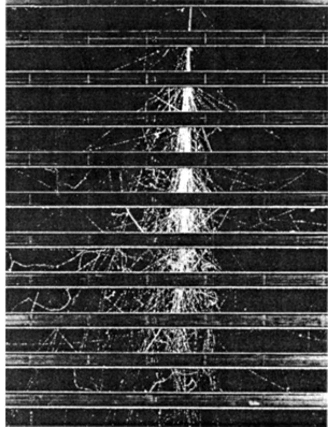


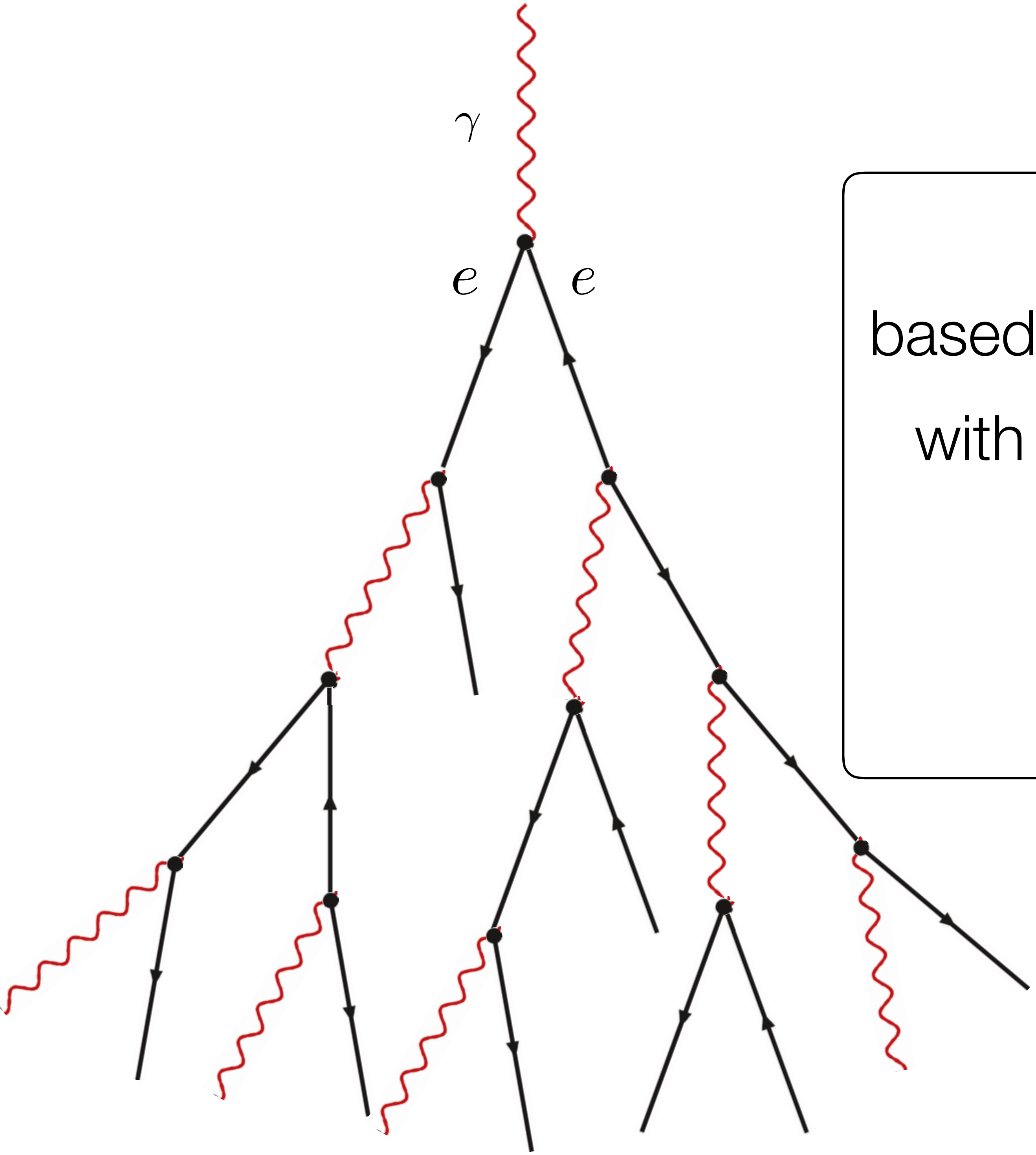
spectra of e± and γ as a function of energy and depth along the shower

bremsstrahlung and pair production populate the shower with e± and γ up to some equilibrium

collision losses and Compton scattering cause energy losses deeply affecting the shower development

electromagnetic showers
cascade equations





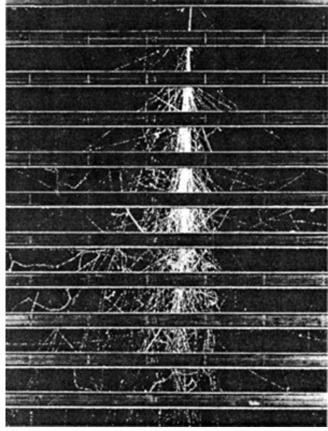
Approximation A

describe an EM shower
based on bremsstrahlung and pair production
with asymptotic (high energy) scale-invariant
splitting functions

neglect energy losses

electromagnetic showers

cascade equations

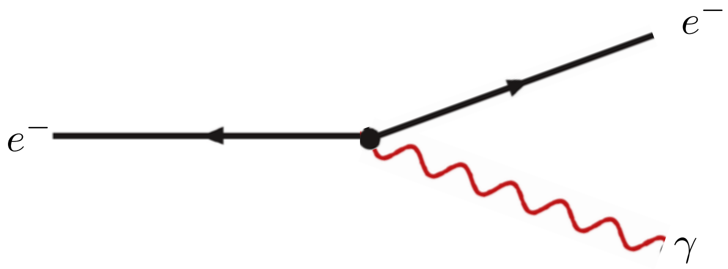


depth
in units of
radiation length

$$t = \frac{X}{\lambda_{rad}}$$

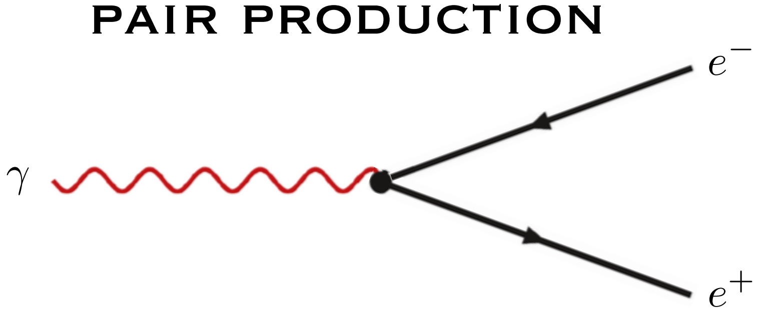
$$\frac{\partial n_e(E,t)}{\partial t} = - \text{LOSS} + \text{GAIN}$$

BREMSSTRAHLUNG



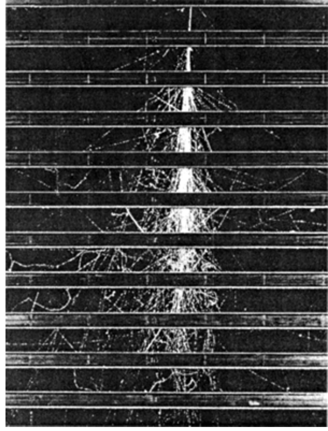
$$\frac{\partial n_\gamma(E,t)}{\partial t} = - \text{LOSS} + \text{GAIN}$$

PAIR PRODUCTION



electromagnetic showers

cascade equations



depth
in units of
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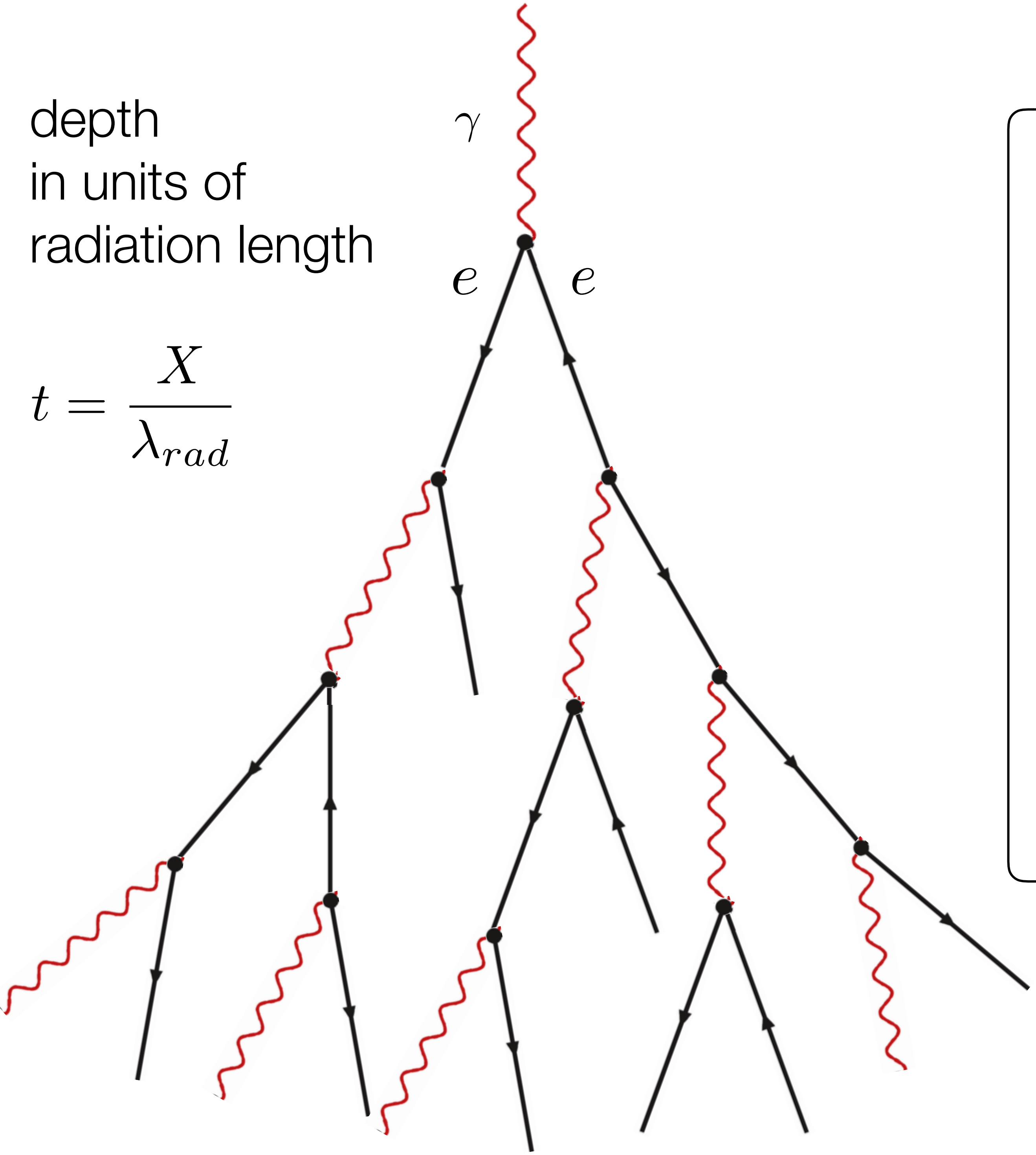
$$t = \frac{X}{\lambda_{rad}}$$

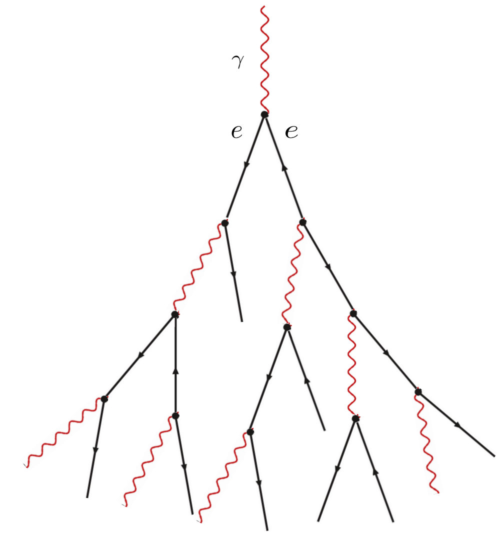


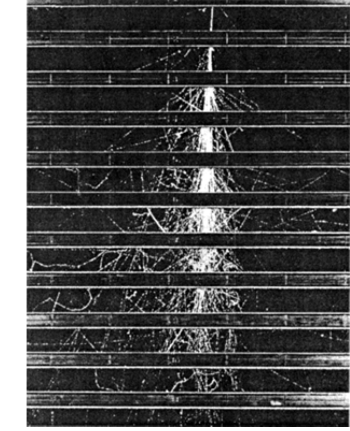
$$\frac{\partial n_e(E,t)}{\partial t} = -(e \to \gamma)_{\text{brems}} + (e \to e)_{\text{brems}} + (\gamma \to e)_{\text{epair}}$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = -(\gamma \to e)_{\text{epair}} + (e \to \gamma)_{\text{brems}}$$

electromagnetic showers

cascade equations

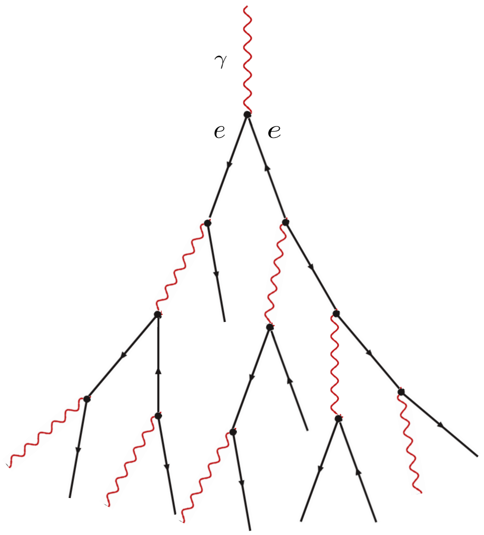


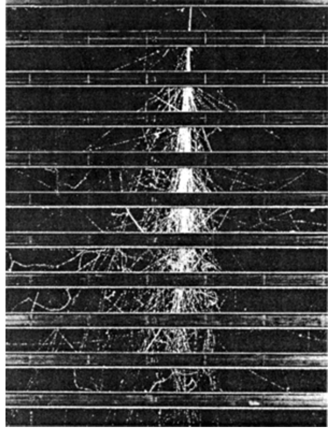


$$\frac{\partial n_e(E,t)}{\partial t} = \begin{aligned} &-(e \to \gamma)_{\mathbf{brems}} \\ &+(e \to e)_{\mathbf{brems}} \\ &+(\gamma \to e)_{\mathbf{epair}} \end{aligned}$$

$$\begin{aligned} \frac{\partial n_e(E,t)}{\partial t} = \quad & -n_e(E,t) \int_0^1 dv \varphi(v) \qquad\qquad -\frac{n_e(E,t)}{\lambda_{rad}} \\ & + \int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E-(1-v)E'] \\ & + \int_E^\infty dE' \int_0^1 du\, n_\gamma(E',t)\, \psi(u)\, \delta[E-uE'] \end{aligned}$$

electromagnetic showers
cascade equations





$$\frac{\partial n_e(E,t)}{\partial t} = \begin{aligned} & -(e \to \gamma)_{\text{brems}} \\ & +(e \to e)_{\text{brems}} \\ & +(\gamma \to e)_{\text{epair}} \end{aligned}$$

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv\varphi(v)\left[n_e(E,t) - \frac{1}{1-v}n_e\left(\frac{E}{1-v},t\right)\right]$$

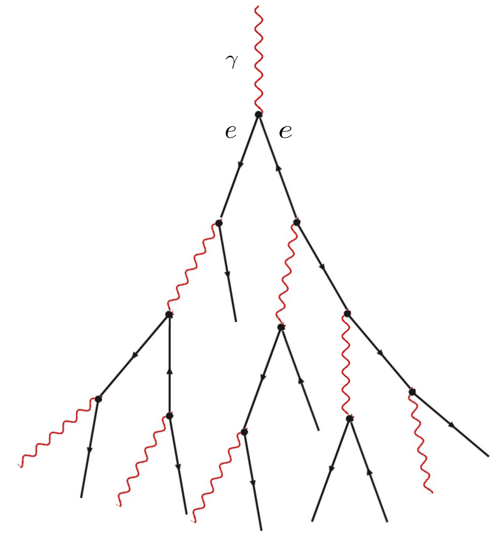
brems

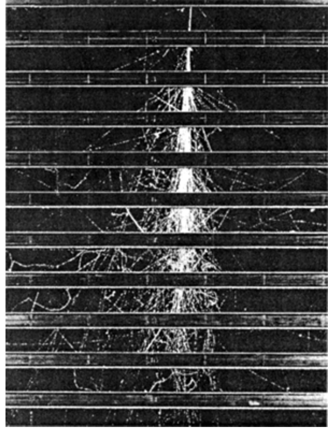
$$+2\int_0^1 \frac{du}{u}\,\psi(u)\,n_\gamma\left(\frac{E}{u},t\right)$$

epair

electromagnetic showers

cascade equations





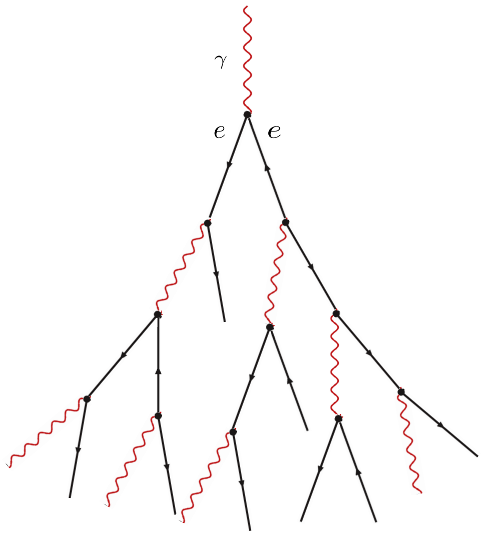
$$\frac{\partial n_\gamma(E,t)}{\partial t} = -(\gamma \to e)_{\text{epair}} + (e \to \gamma)_{\text{brems}}$$

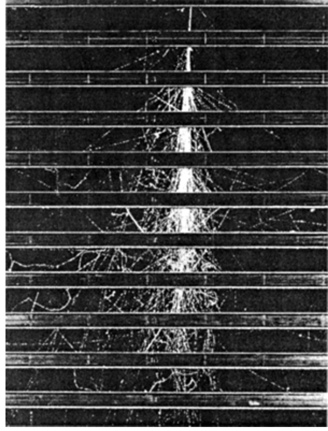
$$\frac{\partial n_\gamma(E,t)}{\partial t} = -n_\gamma(E,t) \int_0^1 du\, \psi(u) + \int_E^\infty dE' \int_0^1 dv\, n_e(E',t)\, \varphi(v)\, \delta[E - vE']$$

$$-\sigma_0 \frac{n_\gamma(E,t)}{\lambda_{pair}}$$

electromagnetic showers

cascade equations





$$\frac{\partial n_\gamma(E,t)}{\partial t} = \quad -(\gamma \to e)_{\text{epair}}$$
$$+(e \to \gamma)_{\text{brems}}$$

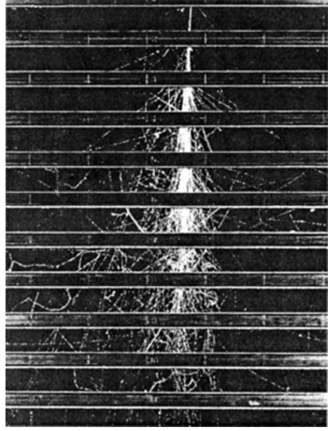
$$\frac{\partial n_\gamma(E,t)}{\partial t} = \quad -\sigma_0 \, n_\gamma(E,t)$$
$$+\int_0^1 \frac{dv}{v}\,\varphi(v)\,n_e\left(\frac{E}{v},t\right)$$

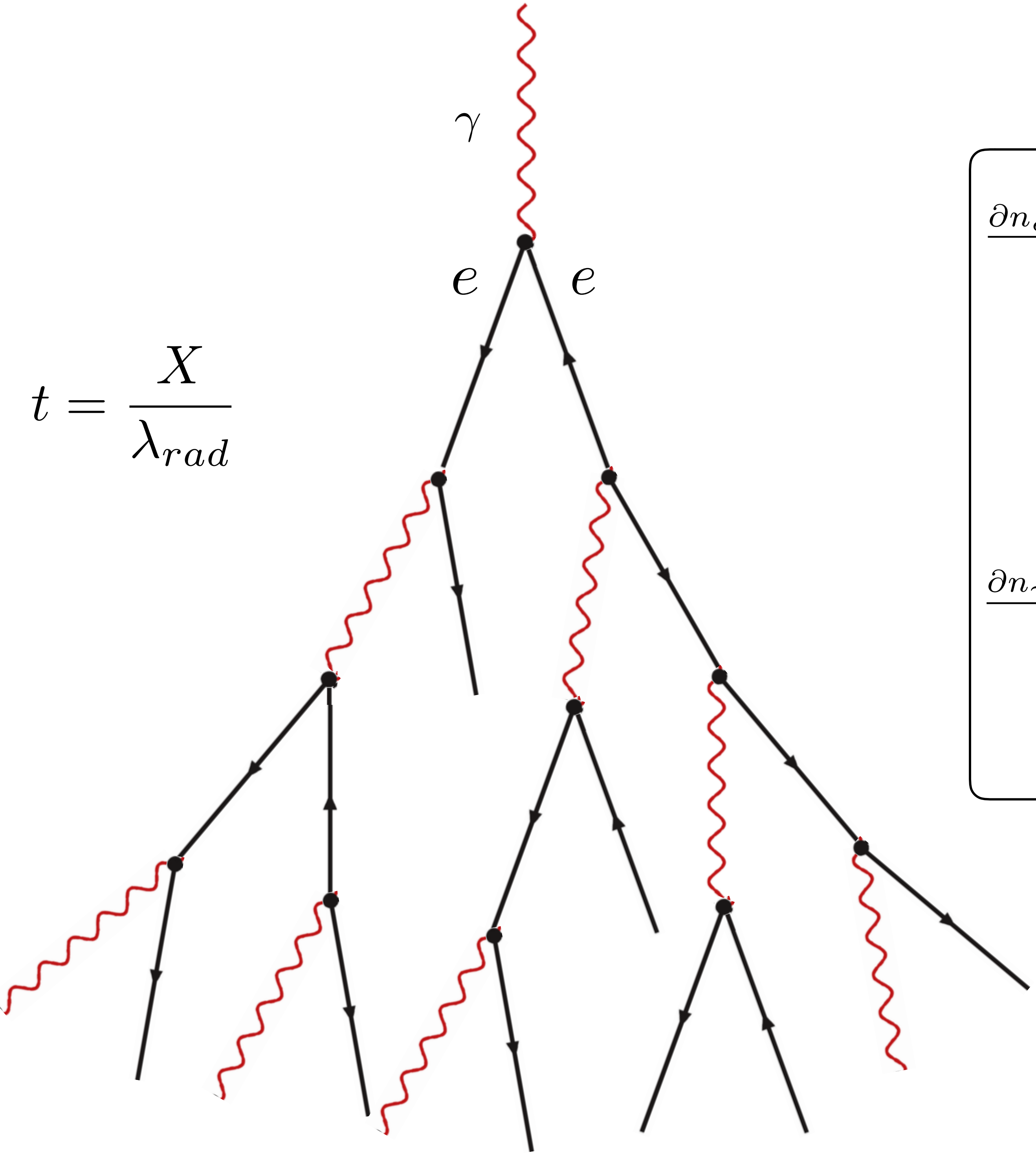
epair

brems

electromagnetic showers

cascade equations





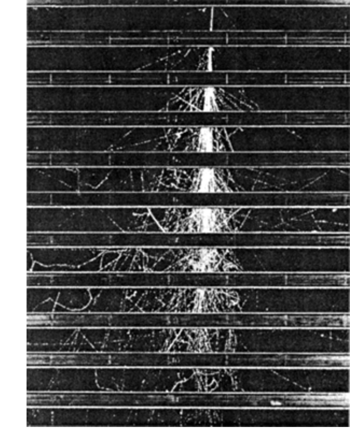
$$t = \frac{X}{\lambda_{rad}}$$

Approximation A

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv \varphi(v) \left[n_e(E,t) - \frac{1}{1-v} n_e\left(\frac{E}{1-v}, t\right)\right] + 2\int_0^1 \frac{du}{u}\, \psi(u)\, n_\gamma\left(\frac{E}{u}, t\right)$$

$$\frac{\partial n_\gamma(E,t)}{\partial t} = -\sigma_0\, n_\gamma(E,t) + \int_0^1 \frac{dv}{v}\, \varphi(v)\, n_e\left(\frac{E}{v}, t\right)$$

electromagnetic showers
cascade equations



Approximation A

$$
\begin{aligned}
\frac{\partial n_e(E,t)}{\partial t} &= -\int_0^1 dv\varphi(v)\left[n_e(E,t) - \frac{1}{1-v} n_e\left(\frac{E}{1-v}, t\right)\right] \\
&\quad +2\int_0^1 \frac{du}{u}\,\psi(u)\,n_\gamma\left(\frac{E}{u}, t\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial n_\gamma(E,t)}{\partial t} &= -\sigma_0\, n_\gamma(E,t) \\
&\quad +\int_0^1 \frac{dv}{v}\,\varphi(v)\,n_e\left(\frac{E}{v}, t\right)
\end{aligned}
$$

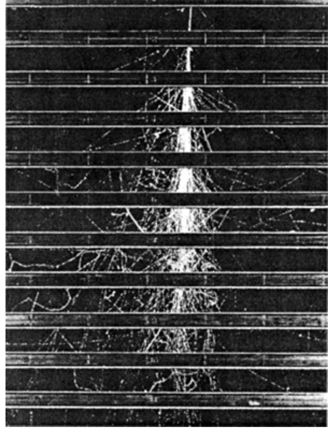
re-write the equations using
Mellin transform of spectra functions

$$
\tilde{n}(s) = \int_0^\infty E^s\, n(E)\, dE
$$

$$
\frac{\partial \tilde{n}_e(s,t)}{\partial t} = A(s)\,\tilde{n}_e(s,t) + B(s)\,\tilde{n}_\gamma(s,t)
$$

$$
\frac{\partial \tilde{n}_\gamma(s,t)}{\partial t} = C(s)\,\tilde{n}_e(s,t) - \sigma_0\,\tilde{n}_\gamma(s,t)
$$

electromagnetic showers
spectrum weighted moments of splitting functions



$$A(s) = \int_0^1 dv\, \varphi(v)[1-(1-v)^s]$$
$$= \left(\frac{4}{3}+2b\right)\left(\frac{\Gamma'(1+s)}{\Gamma(1+s)}+\gamma\right)+\frac{6[7+5s+12b(2+s)]}{s(1+s)(2+s)}$$

$\gamma = 0.5772$ Euler's constant

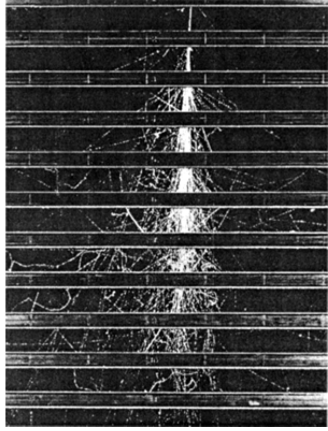
$$B(s) = 2\int_0^1 du\, u^s\, \psi(u)$$
$$= \frac{2[14+11s+3s^2-6b(1+s)]}{3(1+s)(2+s)(3+s)}$$

$$C(s) = \int_0^1 dv\, v^s\, \varphi(v)$$
$$= \frac{8+7s+3s^2+6b(2+s)}{3s(2+3s+s^2)}$$

$$\sigma_0 = \int_0^1 du\, \psi(u) = \frac{7}{9}-\frac{b}{3}$$

electromagnetic showers

cascade equations - matrix formalism



Approximation A

$$\frac{\partial \tilde{n}_e(s,t)}{\partial t} = A(s)\,\tilde{n}_e(s,t) + B(s)\,\tilde{n}_\gamma(s,t)$$

$$\frac{\partial \tilde{n}_\gamma(s,t)}{\partial t} = C(s)\,\tilde{n}_e(s,t) - \sigma_0\,\tilde{n}_\gamma(s,t)$$

we now have a linear system of differential equations

initial conditions:

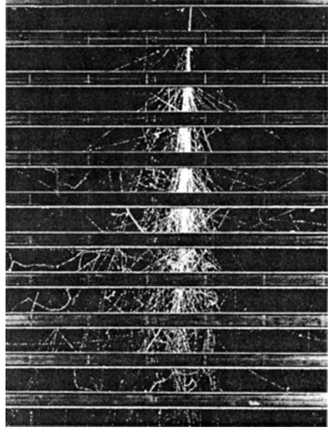
$$n_\gamma(E,0) = \delta(E - E_0) \qquad n_e(E,0) = 0$$

$$\tilde{n}_\gamma(s,0) = E_0^s \qquad \tilde{n}_e(s,0) = 0$$

photon-initiated cascade OR electron-initiated cascade

electromagnetic showers

cascade equations - matrix formalism



Approximation A

$$\frac{\partial \tilde{n}_e(s,t)}{\partial t} = A(s)\,\tilde{n}_e(s,t) + B(s)\,\tilde{n}_\gamma(s,t)$$

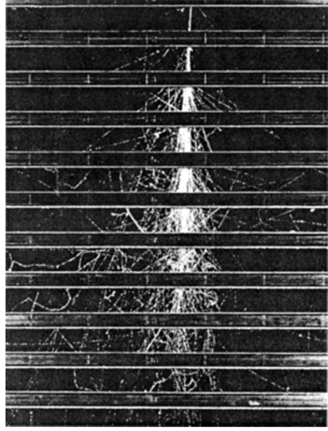
$$\frac{\partial \tilde{n}_\gamma(s,t)}{\partial t} = C(s)\,\tilde{n}_e(s,t) - \sigma_0\,\tilde{n}_\gamma(s,t)$$

we now have a linear system of differential equations

$$\tilde{n}_e(s,t) = \frac{E_0^s}{\lambda_1(s) - \lambda_2(s)} \left[(\sigma_0 + \lambda_1(s))\, e^{\lambda_1(s)\,t} - (\sigma_0 + \lambda_2(s))\, e^{\lambda_2(s)\,t}\right]$$

$$\tilde{n}_\gamma(s,t) = \frac{C(s)\,E_0^s}{\lambda_1(s) - \lambda_2(s)} \left[e^{\lambda_1(s)\,t} - e^{\lambda_2(s)\,t}\right]$$

electromagnetic showers
cascade equations - elementary solutions



Approximation A

$$\tilde{n}_e(s,t) = \frac{E_0^s}{\lambda_1(s) - \lambda_2(s)} \left[(\sigma_0 + \lambda_1(s))\, e^{\lambda_1(s)\,t} - (\sigma_0 + \lambda_2(s))\, e^{\lambda_2(s)\,t} \right]$$

$$\tilde{n}_\gamma(s,t) = \frac{C(s)\, E_0^s}{\lambda_1(s) - \lambda_2(s)} \left[e^{\lambda_1(s)\,t} - e^{\lambda_2(s)\,t} \right]$$

$$\lambda_{1,2}(s) = -\frac{1}{2}(A(s) + \sigma_0) \pm \frac{1}{2}\sqrt{(A(s) - \sigma_0)^2 + 4B(s)C(s)}$$

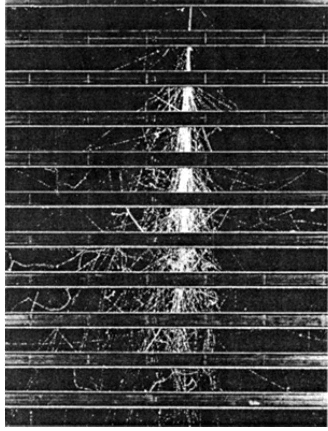
eigenvalues

$$r_\gamma^{(1,2)}(s) = \frac{C(s)}{\sigma_0 + \lambda_{1,2}(s)} = \frac{\text{photons}}{\text{electrons}}$$

eigenvectors

electromagnetic showers

cascade equations - shower development

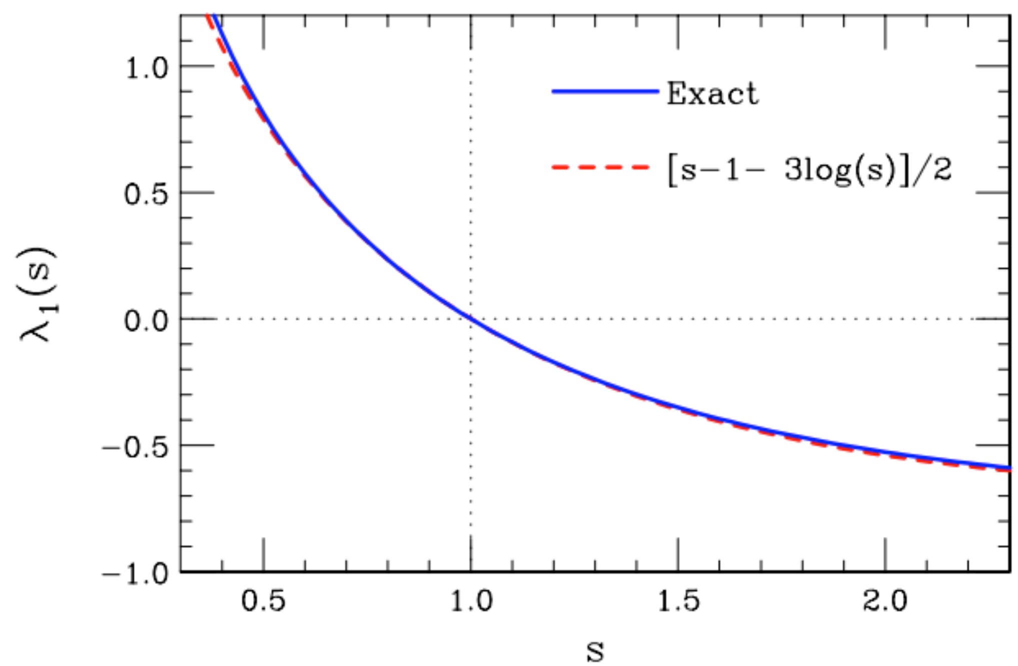


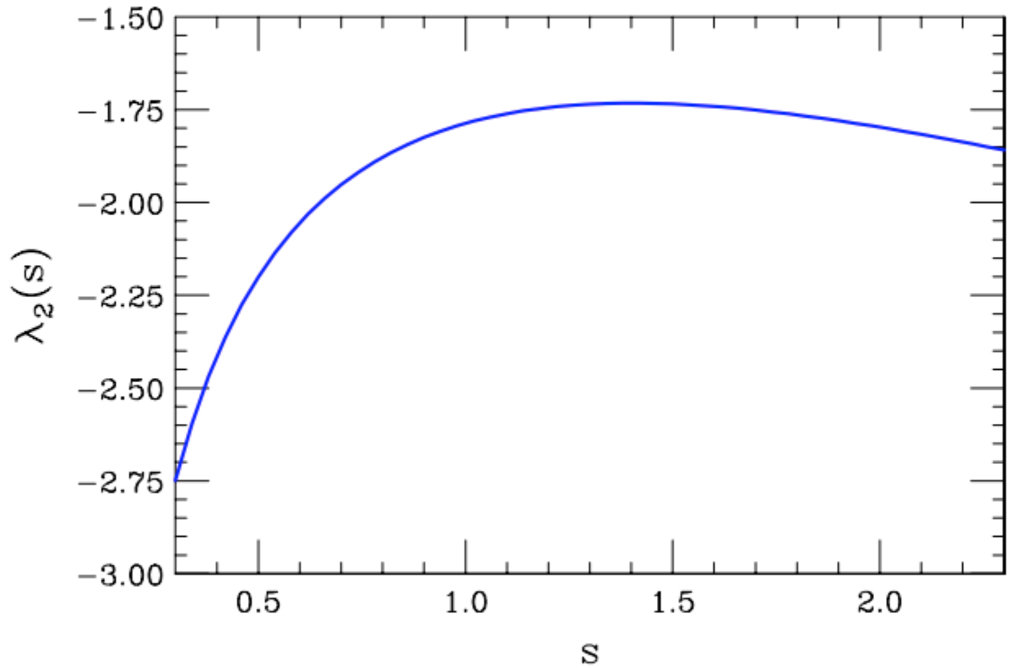
Approximation A

Greisen

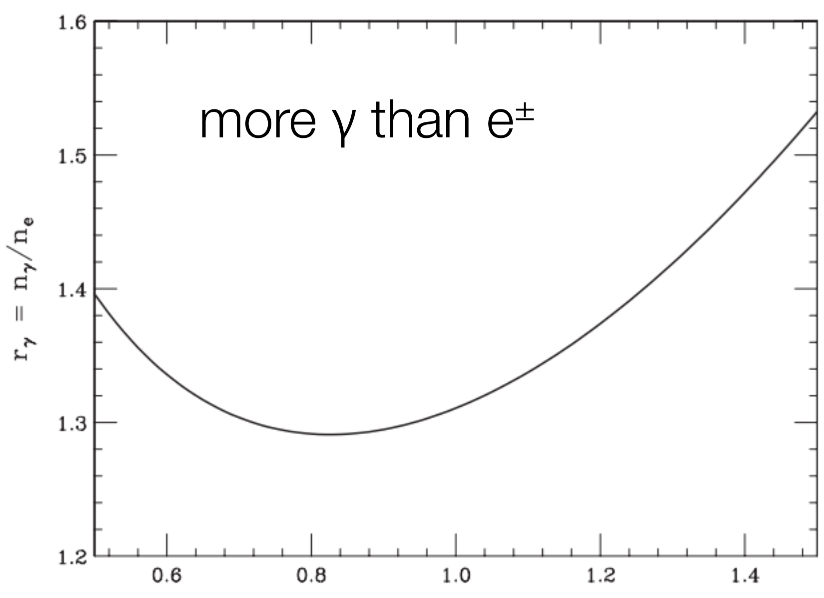
$$\lambda_{1,2}(s) = -\frac{1}{2}(A(s)+\sigma_0) \pm \frac{1}{2}\sqrt{(A(s)-\sigma_0)^2 + 4B(s)C(s)}$$

$$\lambda_1(s) \approx \bar{\lambda}_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$$





$$|\lambda_2(s)| > |\lambda_1(s)|$$



$$r_\gamma^{(1,2)}(s) = \frac{C(s)}{\sigma_0 + \lambda_{1,2}(s)}$$

independently of initial particle type

spectra reach **asymptotic γ/e** ratio

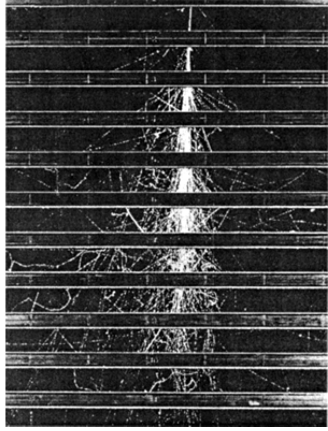
at **$t \sim |\lambda_2(s)|^{-1}$**

then evolves as **$e^{\lambda_1(s)t}$**

maintaining a constant equilibrium ratio **$r_\gamma^1(s)$**

electromagnetic showers

cascade equations - Mellin Transform



Approximation A

inverse Mellin transforms

$$n(E) = \frac{1}{2\pi i}\int_C ds\, E^{-(s+1)}\, \tilde{n}(s)$$

$$\frac{1}{2\pi i}\int_{s_0 - i\infty}^{s_0 + i\infty} ds\, E^{-(s+1)}\, \tilde{n}(s)$$

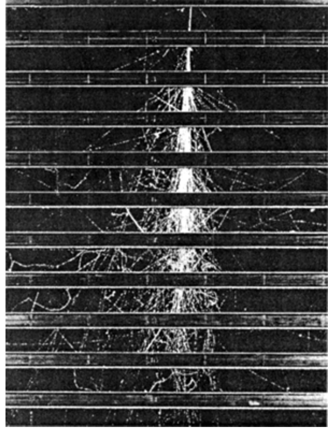
this integral **cannot** be solved analytically but can be solved numerically

or use the ***saddle point approximation***

$$n_\gamma(E,t) = \frac{1}{2\pi i}\int_C E^{-(s+1)}\, \frac{C(s)\, E_0^s}{\lambda_1(s) - \lambda_2(s)}\, e^{\lambda_1(s)\, t}\, ds \qquad t \gg |\lambda_2(s)|^{-1}$$

electromagnetic showers

cascade equations - saddle point approximation



Approximation A

$$n_\gamma(E,t) = \frac{1}{2\pi i} \int_C E^{-(s+1)} \frac{C(s)\, E_0^s}{\lambda_1(s) - \lambda_2(s)} e^{\lambda_1(s)\, t}\, ds$$

$t \gg |\lambda_2(s)|^{-1}$

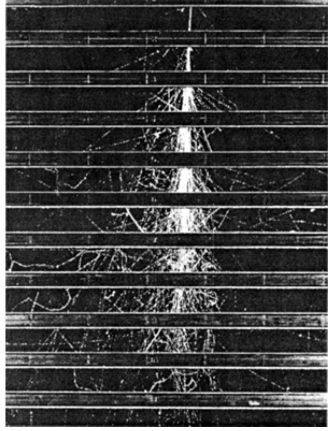
saddle point approximation: separate terms with strong s-dependence

$$n_\gamma(E,t) = \frac{1}{2\pi i} \int_C \frac{1}{E} \left(\frac{E_0}{E}\right)^s \frac{C(s)}{\lambda_1(s) - \lambda_2(s)} e^{\lambda_1(s)\, t}\, ds$$

$$n_\gamma(E,t) = \frac{1}{2\pi i} \frac{1}{E} \int_C \frac{C(s)}{\lambda_1(s) - \lambda_2(s)} e^{\lambda_1(s)\, t + s \ln(E_0/E)}\, ds$$

electromagnetic showers

cascade equations - saddle point approximation



Approximation A

saddle point approximation: separate terms with strong s-dependence

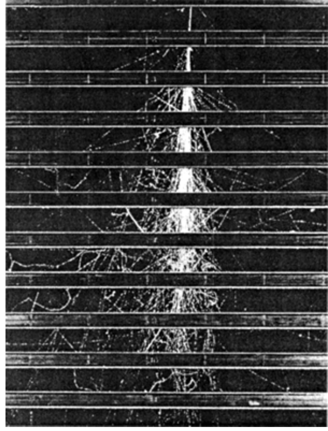
$$n_\gamma(E,t) = \frac{1}{2\pi i}\,\frac{1}{E}\int_C \frac{C(s)}{\lambda_1(s)-\lambda_2(s)}\, e^{\lambda_1(s)\,t + s\ln(E_0/E)}\, ds$$

$$n_\gamma(E,t) = \frac{1}{2\pi i}\,\frac{1}{E}\int_C \frac{C(s)}{\sqrt{s}\,[\lambda_1(s)-\lambda_2(s)]}\, e^{\lambda_1(s)\,t + s\ln(E_0/E) + 1/2\ln s}\, ds$$

slowly varying
value @saddle point

rapidly varying
find extremum (saddle point)

$$\forall\, f(z) = f(x+iy) \rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$ integrand function has a **saddle point**



electromagnetic showers
cascade equations - approximate solution

Approximation A

$$n_\gamma(E,t) = \frac{1}{2\pi i}\,\frac{1}{E}\int_C \frac{C(s)}{\sqrt{s}\,[\lambda_1(s)-\lambda_2(s)]}\,e^{\lambda_1(s)\,t+s\ln(E_0/E)+1/2\ln s}\,ds$$

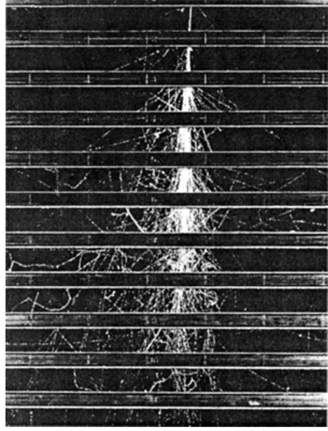
$$\frac{d}{ds}[\lambda_1(s)\,t+s\ln(E_0/E)+1/2\ln s]=0 \qquad \lambda_1'(\bar{s})\,t+\ln(E_0/E)+\frac{1}{2\bar{s}}=0$$

$$n_\gamma(E,t)\approx g(\bar{s})\,\frac{1}{E}\,e^{\lambda_1(\bar{s})\,t+\bar{s}\ln(E_0/E)+1/2\ln\bar{s}}$$

$$\boxed{n_\gamma(E,t)\approx g(\bar{s})\,\frac{1}{E_0}\left(\frac{E}{E_0}\right)^{-(s+1)}e^{\lambda_1(\bar{s})\,t}}$$

electromagnetic showers

cascade equations - approximate solution



Approximation A

$$n_e(E,t) \approx g(\bar{s})\,\frac{1}{E_0}\left(\frac{E}{E_0}\right)^{-(s+1)} e^{\lambda_1(\bar{s})\,t}$$

$$n_\gamma(E,t) \approx g(\bar{s})\,\frac{r_\gamma^{(1)}(\bar{s})}{E_0}\left(\frac{E}{E_0}\right)^{-(s+1)} e^{\lambda_1(\bar{s})\,t}$$

$$r_\gamma^{(1)}(\bar{s}) = \frac{C(\bar{s})}{\sigma_0 + \lambda_1(\bar{s})}$$

shower age defined by the solution to

$$\lambda_1'(\bar{s})\,t + \ln(E_0/E) + \frac{1}{2\bar{s}} = 0$$

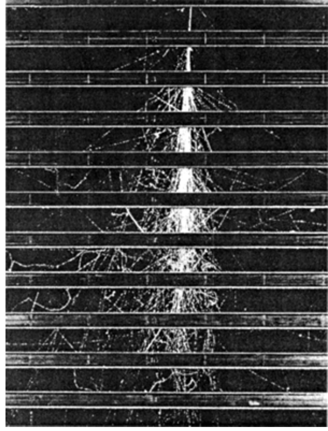
Greisen

$$\lambda_1(s) \approx \bar{\lambda}_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$$

$$\bar{s} \approx \frac{3t}{t + 2\ln\frac{E_0}{E}}$$

electromagnetic showers

cascade equations - approximate solution



Approximation A

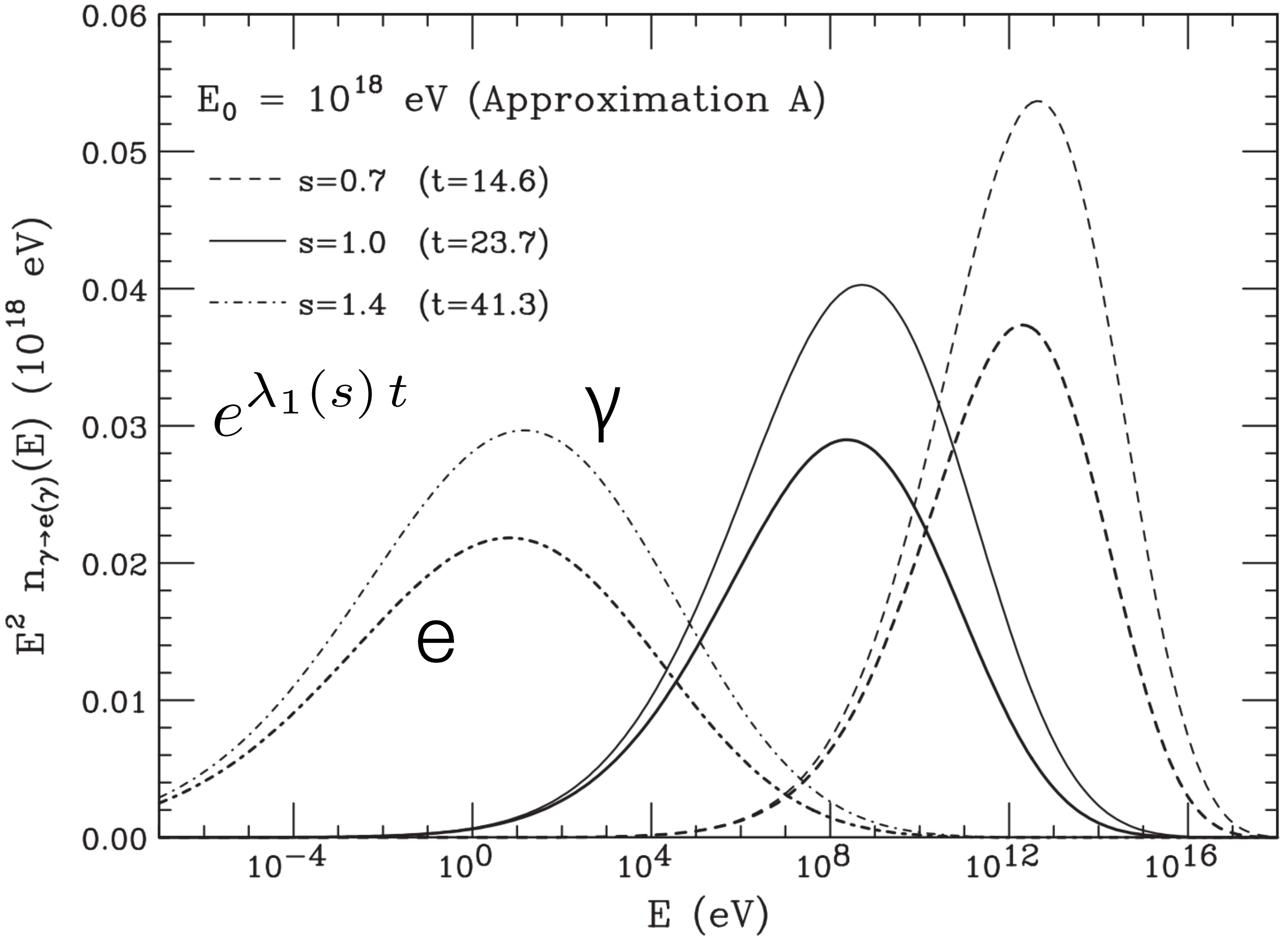
$$n_e(E,t) \approx g(\bar{s})\,\frac{1}{E_0}\,\left(\frac{E}{E_0}\right)^{-(s+1)}\,e^{\lambda_1(\bar{s})\,t}$$

$$n_\gamma(E,t) \approx g(\bar{s})\,\frac{r_\gamma^{(1)}(\bar{s})}{E_0}\,\left(\frac{E}{E_0}\right)^{-(s+1)}\,e^{\lambda_1(\bar{s})\,t}$$

$$\lambda_1(s) \approx \bar{\lambda}_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$$

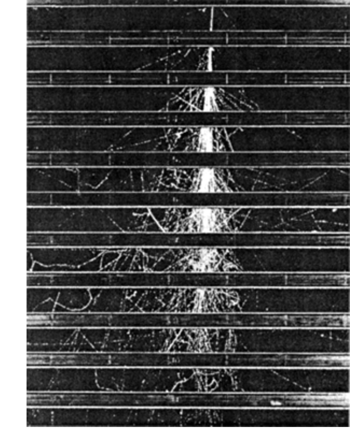
$$\bar{s} \approx \frac{3t}{t + 2\ln\frac{E_0}{E}}$$ **shower age**

$$t_{max}^{[s=1]}\left(\frac{E}{E_0}\right) \simeq \ln\left(\frac{E_0}{E}\right)$$



electromagnetic showers

cascade equations - shower development

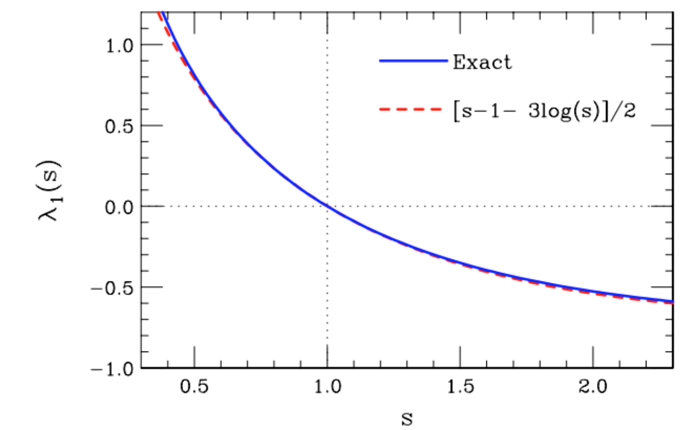


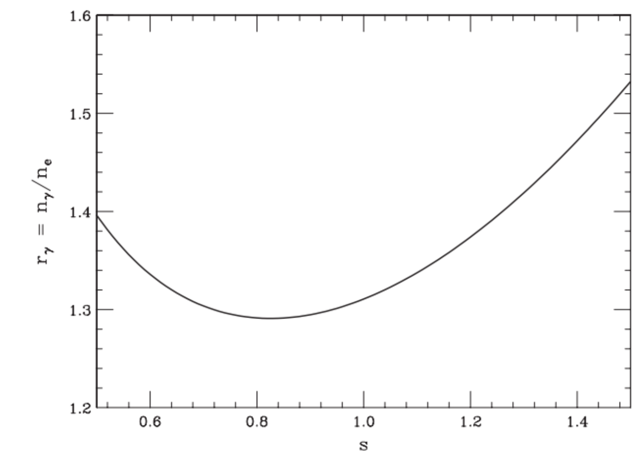
APPROXIMATION A

spectra solution at equilibrium ($t \gg |\lambda_2(s)|^{-1}$)

$$n_e(E,t) = K\,E^{-(s+1)}\,e^{\lambda_1(s)\,t}$$

$$n_\gamma(E,t) = K\,r_\gamma^{(1)}(s)\,E^{-(s+1)}\,e^{\lambda_1(s)\,t}$$





- the spectra remain power law for all values of t

- the only t-independent solution is for s=1 (λ_2=0)
  spectra **do not** depend on cross sections
  shower maximum

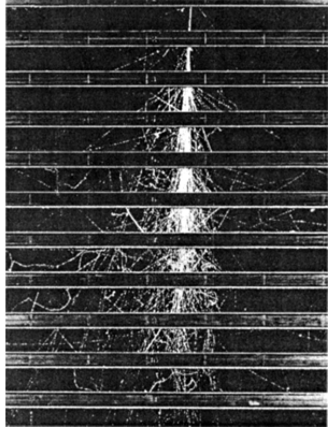
$$n_e(E,t) = K\,E^{-2}$$

$$n_\gamma(E,t) = K\,r_\gamma^{(1)}(s)\,E^{-2}$$

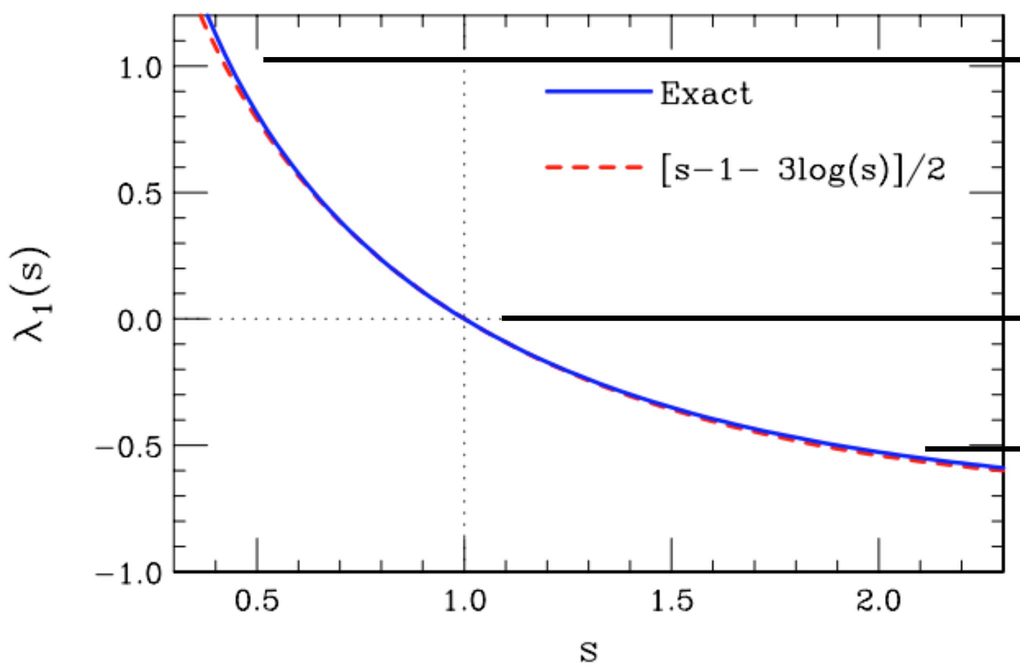
$$\bar{r}_\gamma = r_\gamma^{(1)} \approx 1.31$$

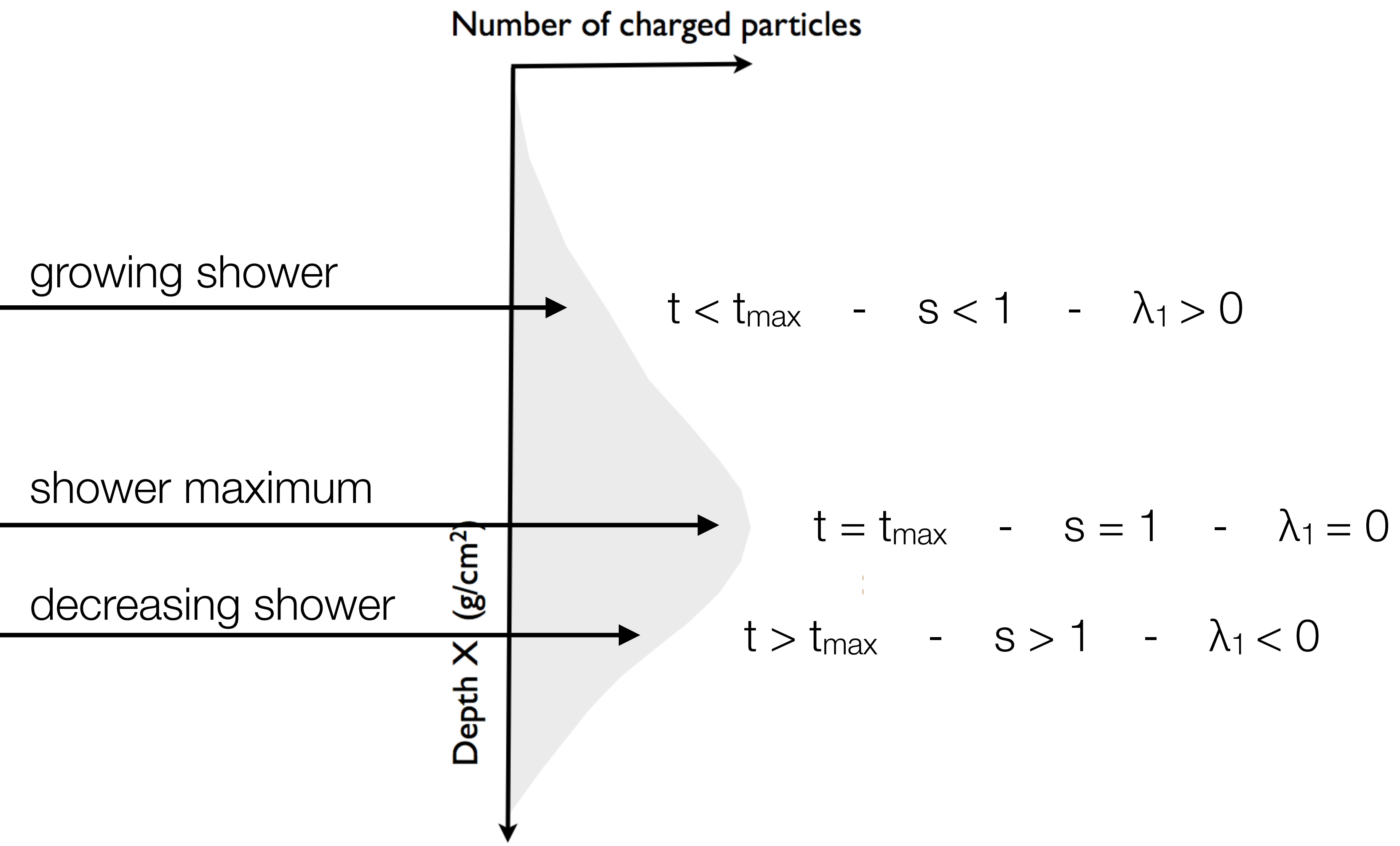
electromagnetic showers

cascade equations - shower development



Approximation A





- **s** evolves with the shower as a function of **t**
- energy spectra evolve from harder to softer

$$n_e(E,t) = K\,E^{-(s+1)}\,e^{\lambda_1(s)\,t}$$

$$n_\gamma(E,t) = K\,r_\gamma^{(1)}(s)\,E^{-(s+1)}\,e^{\lambda_1(s)\,t}$$

electromagnetic showers

generated by a γ or an e

Approximation A

- **s** as ***local slope*** of energy spectra in **t**

$$\bar{s} \approx \frac{3t}{t + 2\ln\frac{E_0}{E}}$$

- local slope changes with E/E_0 & t

$$t_{max}^{[s=1]}\left(\frac{E}{E_0}\right) \simeq \ln\left(\frac{E_0}{E}\right)$$

- s grows monotonically with E/E_0 (steeper spectrum)

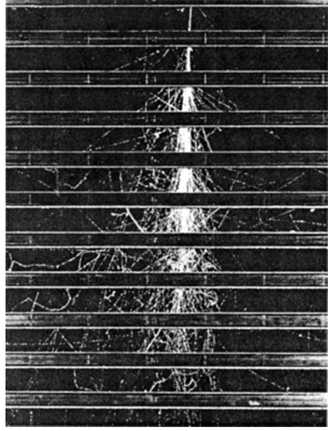
- photon & electron spectra different shapes via $r_\gamma^{(1)}(\bar{s})$

- photons & electrons quickly reach equilibrium independently of what particle initiated the shower

no **energy scaling** reference
@given **t** spectra depend also on $\mathbf{E/E_0}$

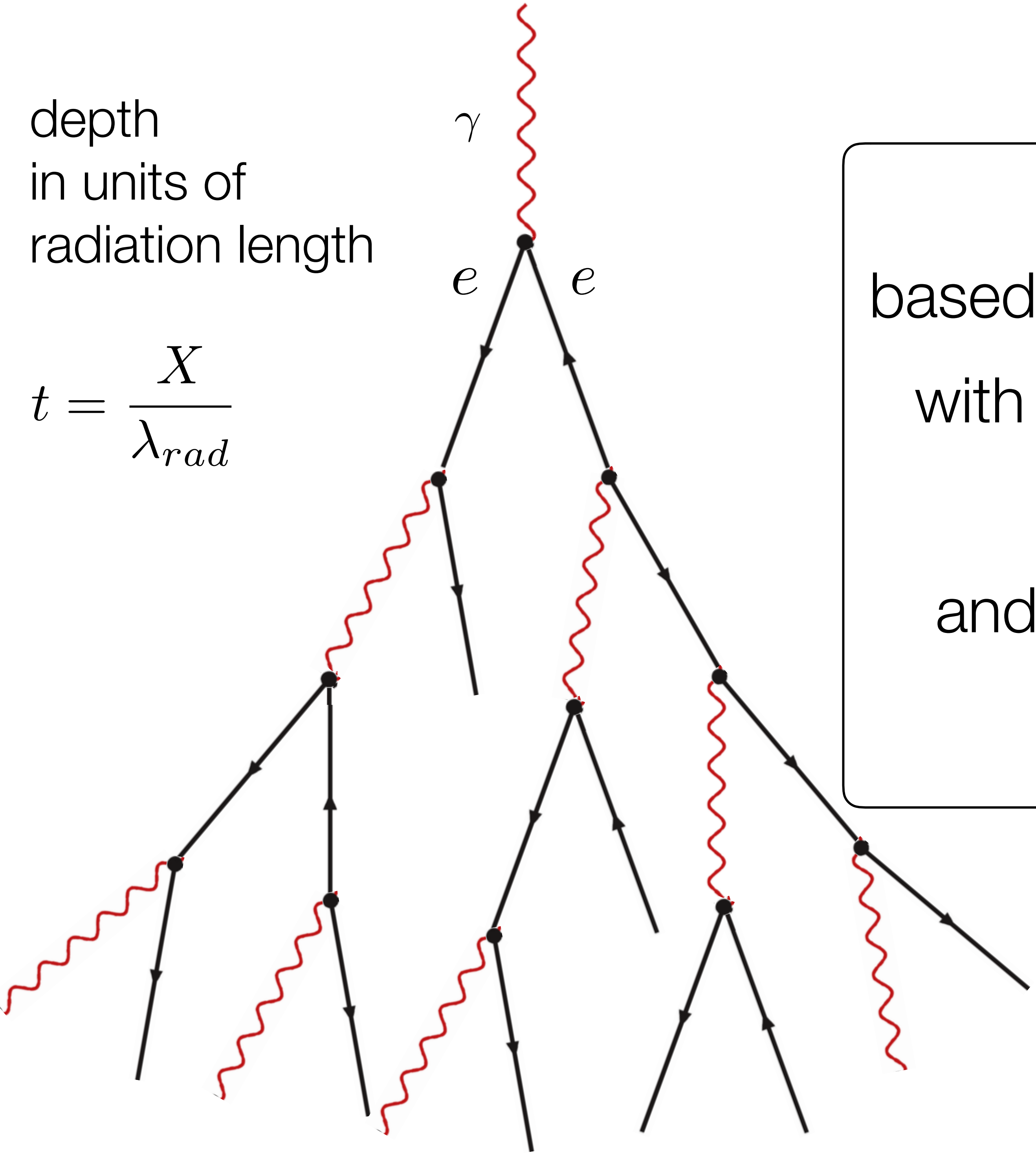
electromagnetic showers

cascade equations



depth
in units of
radiation length

$$t = \frac{X}{\lambda_{rad}}$$



Approximation B

describe an EM shower

based on bremsstrahlung and pair production

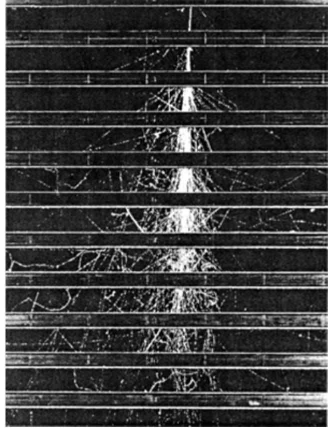
with asymptotic (high energy) scale-invariant

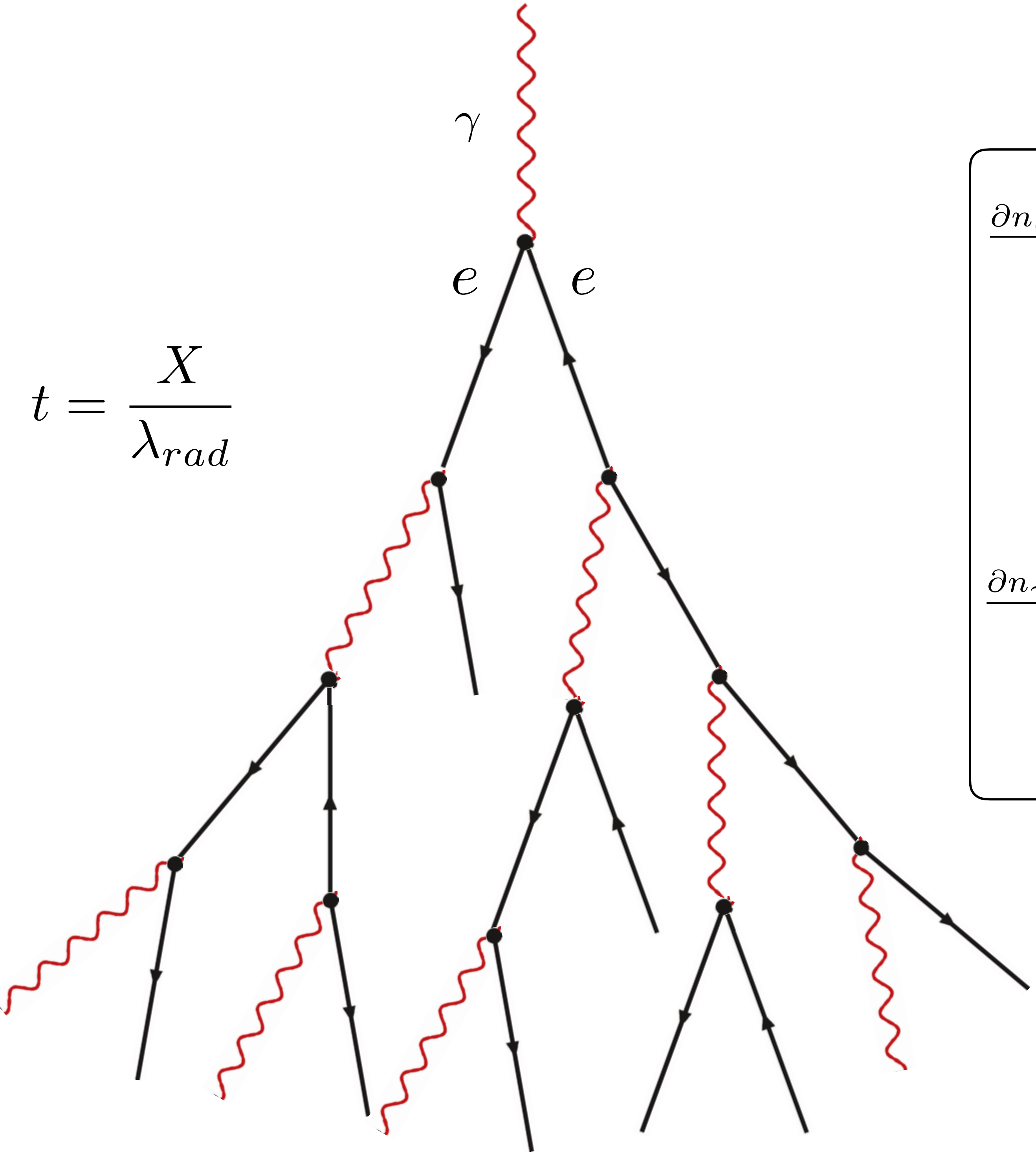
splitting functions

and accounting for collisional energy losses

neglect Compton scattering

electromagnetic showers
cascade equations





$$t = \frac{X}{\lambda_{rad}}$$

Approximation B

$$\frac{\partial n_e(E,t)}{\partial t} = -\int_0^1 dv\varphi(v)\left[n_e(E,t) - \frac{1}{1-v}n_e\left(\frac{E}{1-v},t\right)\right] + 2\int_0^1 \frac{du}{u}\psi(u)\,n_\gamma\left(\frac{E}{u},t\right) + \epsilon\frac{\partial n_e(E,t)}{\partial E}$$

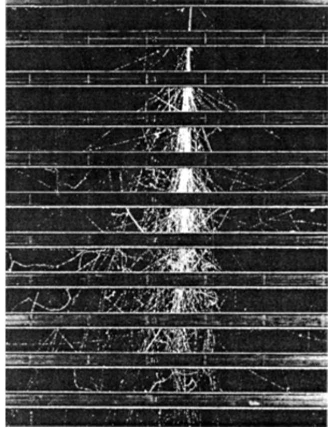
$$\frac{\partial n_\gamma(E,t)}{\partial t} = -\sigma_0\,n_\gamma(E,t) + \int_0^1 \frac{dv}{v}\varphi(v)\,n_e\left(\frac{E}{v},t\right)$$

$$\frac{\partial n(E,t)}{\partial t} = \frac{\partial}{\partial E}[n(E,t)\,\beta(E)]$$

$$-\frac{dE}{dt} = \beta(E)$$

electromagnetic showers

cascade equations - elementary solutions



Approximation B

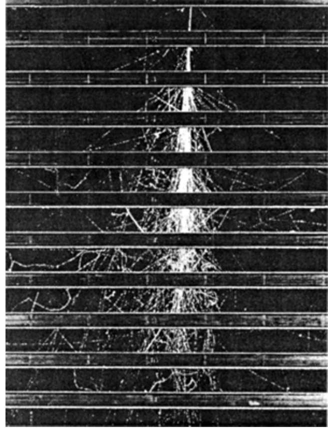
spectra solution at equilibrium ($t \gg |\lambda_2(s)|^{-1}$)

$$n_e(E,t) = K\,E^{-(s+1)}\,e^{\lambda_1(s)\,t} \times p_1\left(s, \frac{E}{\epsilon}\right)$$

$$n_\gamma(E,t) = K\,r_\gamma^{(1)}(s)\,E^{-(s+1)}\,e^{\lambda_1(s)\,t} \times g_1\left(s, \frac{E}{\epsilon}\right)$$

$$p_1\left(s, \frac{E}{\epsilon}\right) \approx \begin{bmatrix} \left(\frac{E}{\epsilon}\right)^{s+1} & \frac{E}{\epsilon} \ll 1 \\ 1 & \frac{E}{\epsilon} \gg 1 \end{bmatrix} \qquad g_1\left(s, \frac{E}{\epsilon}\right) \approx \begin{bmatrix} \left(\frac{E}{\epsilon}\right)^{s} & \frac{E}{\epsilon} \ll 1 \\ 1 & \frac{E}{\epsilon} \gg 1 \end{bmatrix}$$

electromagnetic showers
cascade equations - energy spectra



Approximation B

spectra solution at equilibrium ($t \gg |\lambda_2(s)|^{-1}$)

$$n_e(E,t) = K\, E^{-(s+1)}\, e^{\lambda_1(s)\,t} \times p_1\left(s, \frac{E}{\epsilon}\right)$$

$$n_\gamma(E,t) = K\, r_\gamma^{(1)}(s)\, E^{-(s+1)}\, e^{\lambda_1(s)\,t} \times g_1\left(s, \frac{E}{\epsilon}\right)$$

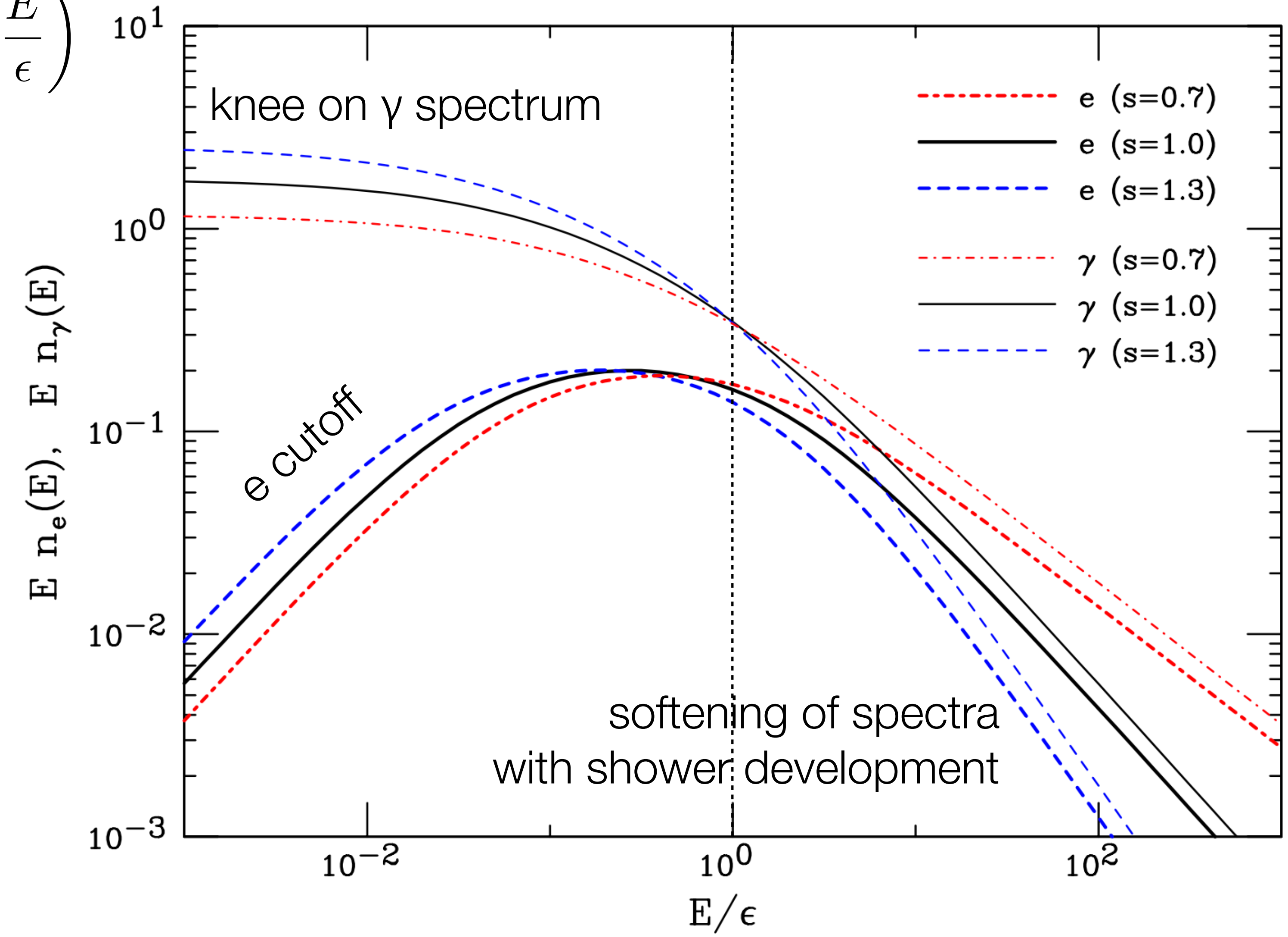
$$p_1\left(s, \frac{E}{\epsilon}\right) \approx \begin{bmatrix} \left(\frac{E}{\epsilon}\right)^{s+1} & \frac{E}{\epsilon} \ll 1 \\ 1 & \frac{E}{\epsilon} \gg 1 \end{bmatrix}$$

$$g_1\left(s, \frac{E}{\epsilon}\right) \approx \begin{bmatrix} \left(\frac{E}{\epsilon}\right)^{s} & \frac{E}{\epsilon} \ll 1 \\ 1 & \frac{E}{\epsilon} \gg 1 \end{bmatrix}$$

no divergence
#electrons well defined

energy scale

critical energy



electromagnetic showers
generated by a γ or an e

Approximation B

- solving $\begin{aligned} n_e(E,t) &= 0 \\ n_\gamma(E,t) &= \delta(E-E_0) \end{aligned}$ or $\begin{aligned} n_e(E,t) &= \delta(E-E_0) \\ n_\gamma(E,t) &= 0 \end{aligned}$

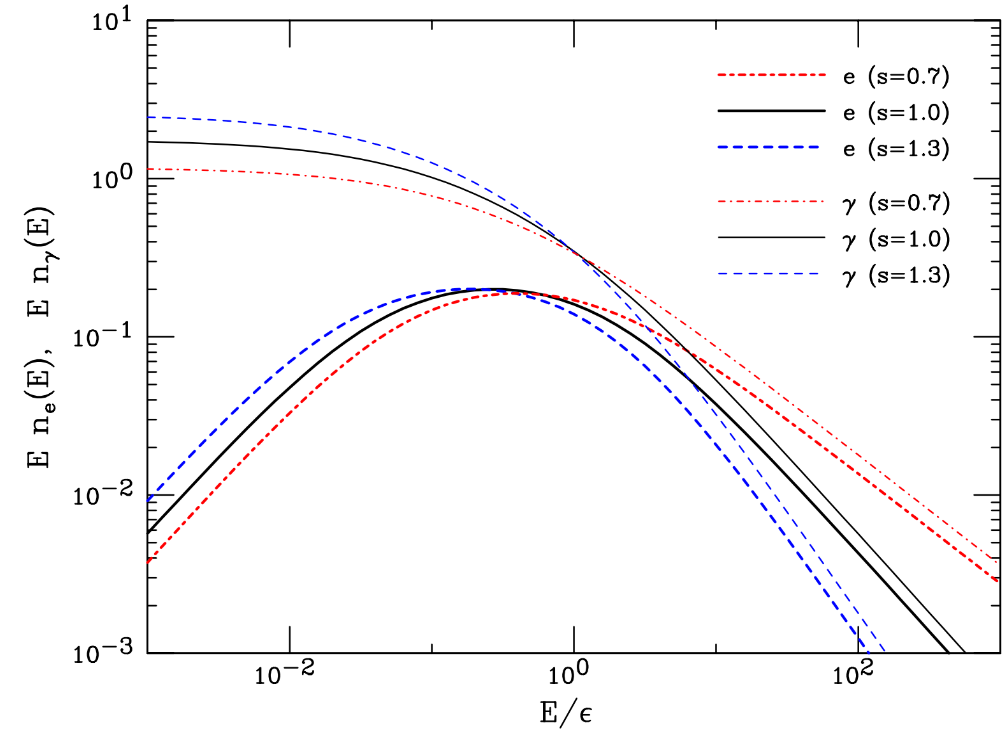
- approximate solution in figure

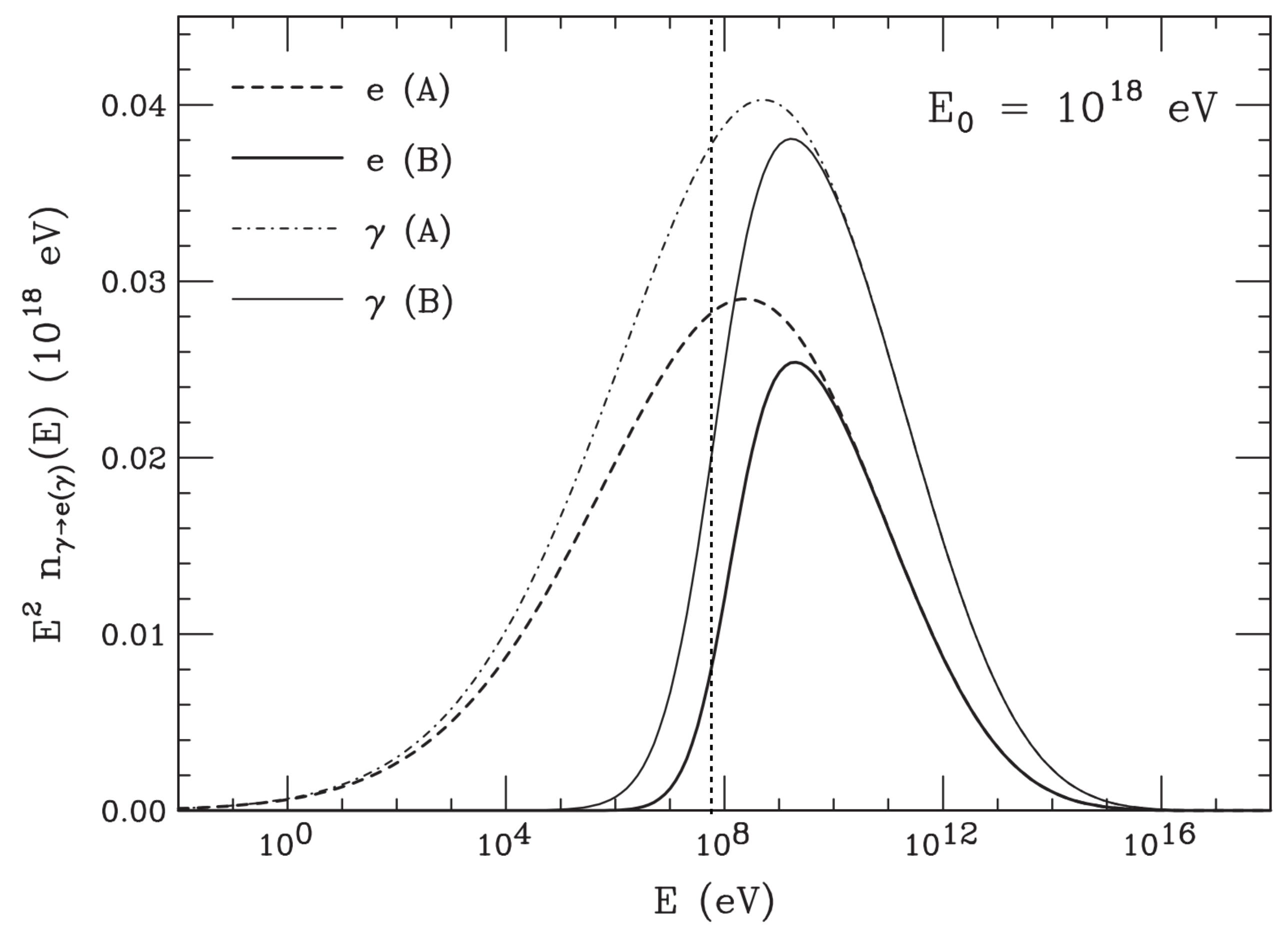
  $t \gg 1$

$$\bar{s}\left(\frac{\epsilon}{E_0}, t\right) \simeq \frac{3t}{t + 2\ln(E_0/\epsilon)}$$

specific s-t mapping

$$t_{max}^{[s=1]} \simeq \ln\left(\frac{E_0}{\epsilon}\right)$$





electromagnetic showers
generated by a γ or an e

Approximation B

$$\bar{s}\left(\frac{\epsilon}{E_0}, t\right) \simeq \frac{3t}{t + 2\ln(E_0/\epsilon)}$$

specific s-t mapping

- **s** as ***global slope*** of energy spectra in **t**
- photon & electron **spectra similar at all t**

$$t_{max}^{[s=1]} \simeq \ln\left(\frac{E_0}{\epsilon}\right)$$

- photons & electrons quickly reach equilibrium independently of what particle initiated the shower

energy scaling provided by collisional loss critical energy
@given **t** spectra are **essentially** well defined

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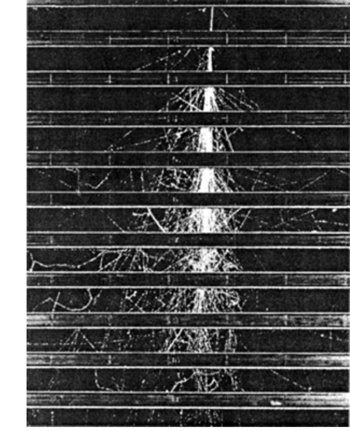
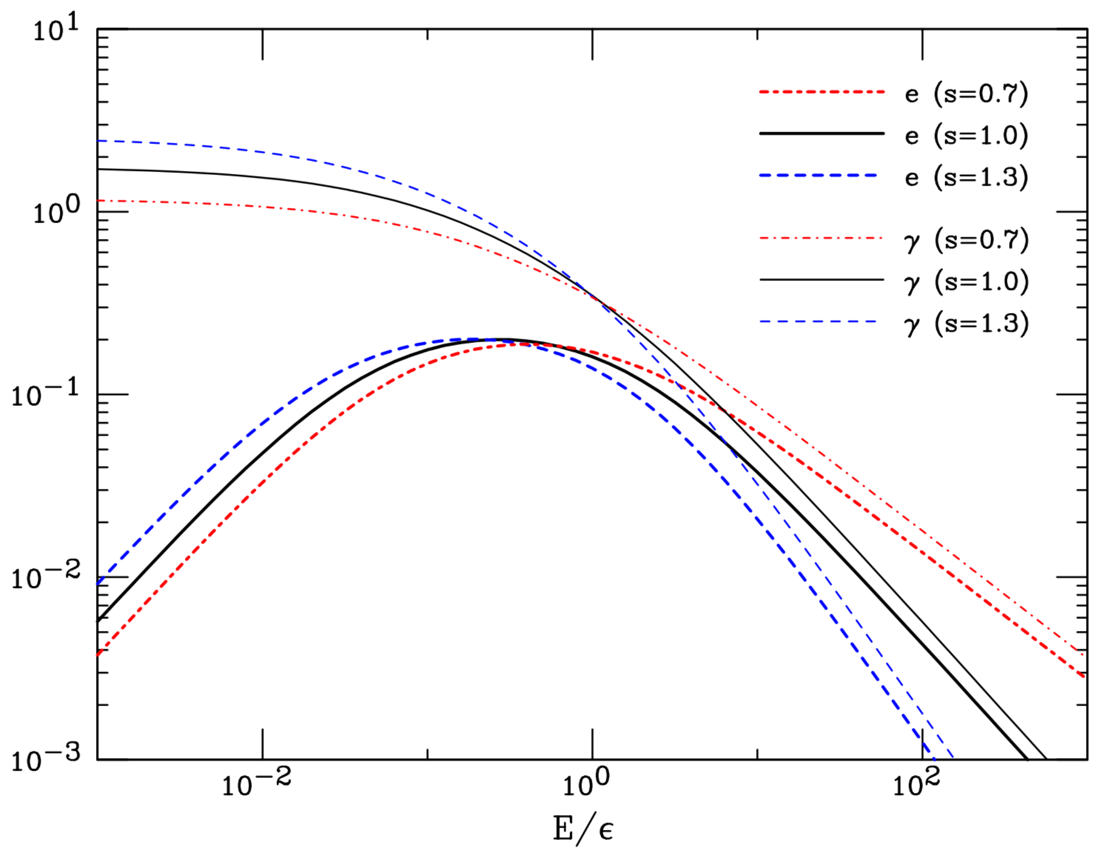
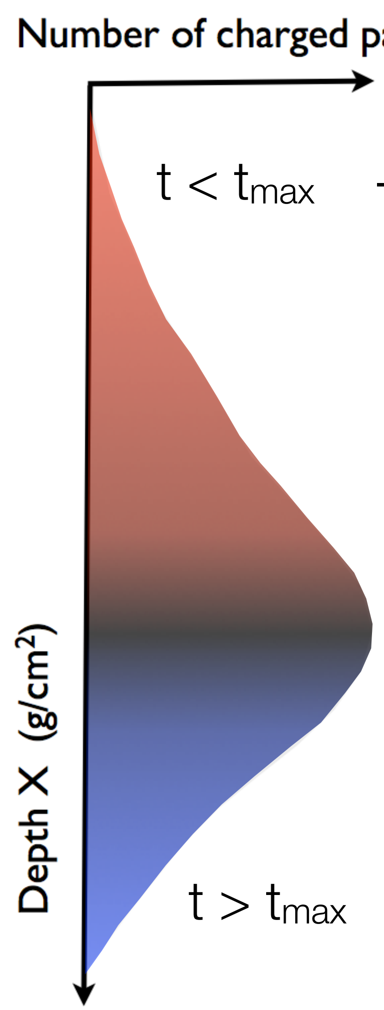
electromagnetic showers

cascade equations - shower development



Approximation B





$E \gg \varepsilon$

$t < t_{max}$ - harder spectra
exponentially increasing normalization

$t = t_{max}$ - E^{-2}
brems & epair redistribute
energy but nothing changes

$t > t_{max}$ - softer spectra
exponentially decreasing normalization

electromagnetic showers

age of a shower

- **age** of a shower related to ***similarity*** of all showers to each other **@maximum**
- **@maximum "*most*"** photons & electrons have same shape and relative norm.

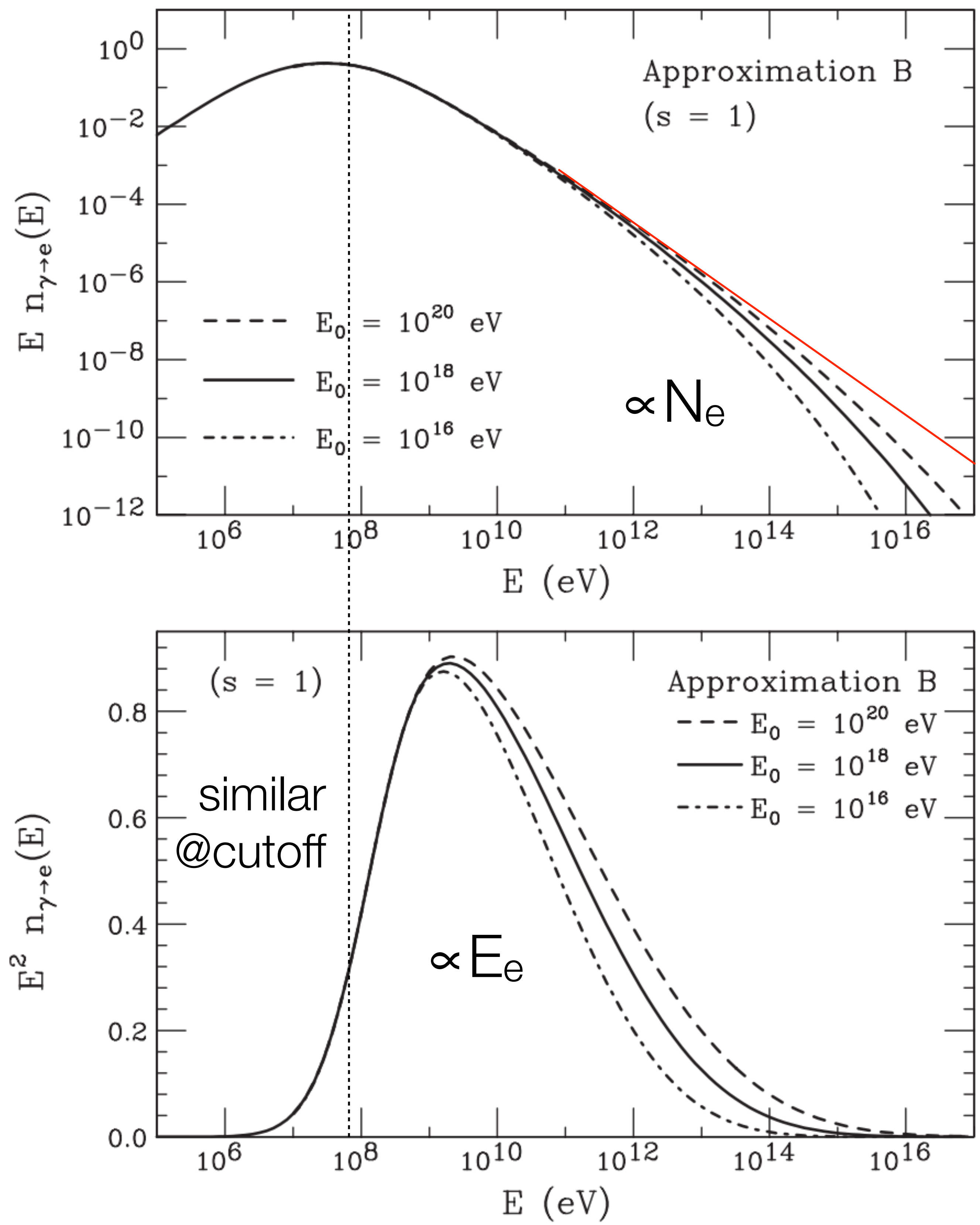
- maximum of shower is where derivative of N(t) vanishes $s = \lambda_1^{-1}\left(\frac{1}{N(t)}\frac{dN(t)}{dt}\right)$
- λ_1^{-1} is the inverse of $\lambda_1(s)$
- in approximation B the age corresponds to $$s = \frac{3t}{t + 2\ln(E_0/\epsilon)}$$
    - with approximate depth @shower maximum $$t_{max}^{[s=1]} \simeq \ln\left(\frac{E_0}{\epsilon}\right)$$

electromagnetic showers

age of a shower

- showers generated by different primaries but with **same age**, have *essentially* the same spectral shape
- in showers of **same age** *"most"* of the particles have same spectra
- **however** distributions at higher energies are **different**
- at high energy t-evolution of showers are not uniquely defined by age, but depends on primary energy (**approximation A**)

@shower maximum



electromagnetic showers

Greisen profile - average development of EM showers

$$N_e(E_{min}, t) = \int_{E_{min}}^{E_{max}} dE\, n_e(E, t)$$

shower size

$$N_e(E_{min}, t) \approx K \left(\frac{E_{min}}{E_0}\right)^{-s} e^{\lambda_1(s)t}$$

$$\epsilon < E < E_0$$

$$\frac{dN(t)}{dt} \simeq \lambda(s)\, N(t)$$

$$\frac{dN(t)}{dt} = \left[\lambda(s) + [\lambda_1'(s)t + \ln(E_0/E_{min})]\, \frac{ds}{dt}\right] N(t)$$

$$\lambda_1'(s)t + \ln(E_0/E_{min}) = 0$$

electromagnetic showers

Greisen profile - **average** development of EM showers

$$\lambda_1'(s)t + \ln(E_0/E_{min}) = 0$$

constrain on the evolution of the shower

$$\frac{1}{2}\left(1 - \frac{3}{s}\right)t + \ln\left(\frac{E_0}{E_{min}}\right) = 0$$

Greisen

$$\lambda_1(s) \approx \bar{\lambda}_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$$

$$s(t) = \frac{3t}{t + 2\ln(E_0/E_{min})}$$

$$t = t_{max}(s = 1) \rightarrow t_{max} = \ln\left(\frac{E_0}{E_{min}}\right)$$

exact
in current approximation

electromagnetic showers

Greisen profile - average development of EM showers

$$\frac{dN(t)}{dt} = \lambda(s)\,N(t)$$

$$\lambda_1(s) \approx \bar{\lambda}_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$$

$$s(t) = \frac{3t}{t + 2\ln(E_0/E_{min})}$$

$$\frac{dN(t)}{dt} = \frac{1}{2}\left[\frac{3t}{t+2t_{max}} - 1 - 3\ln\left(\frac{3t}{t+2t_{max}}\right)\right]N(t)$$

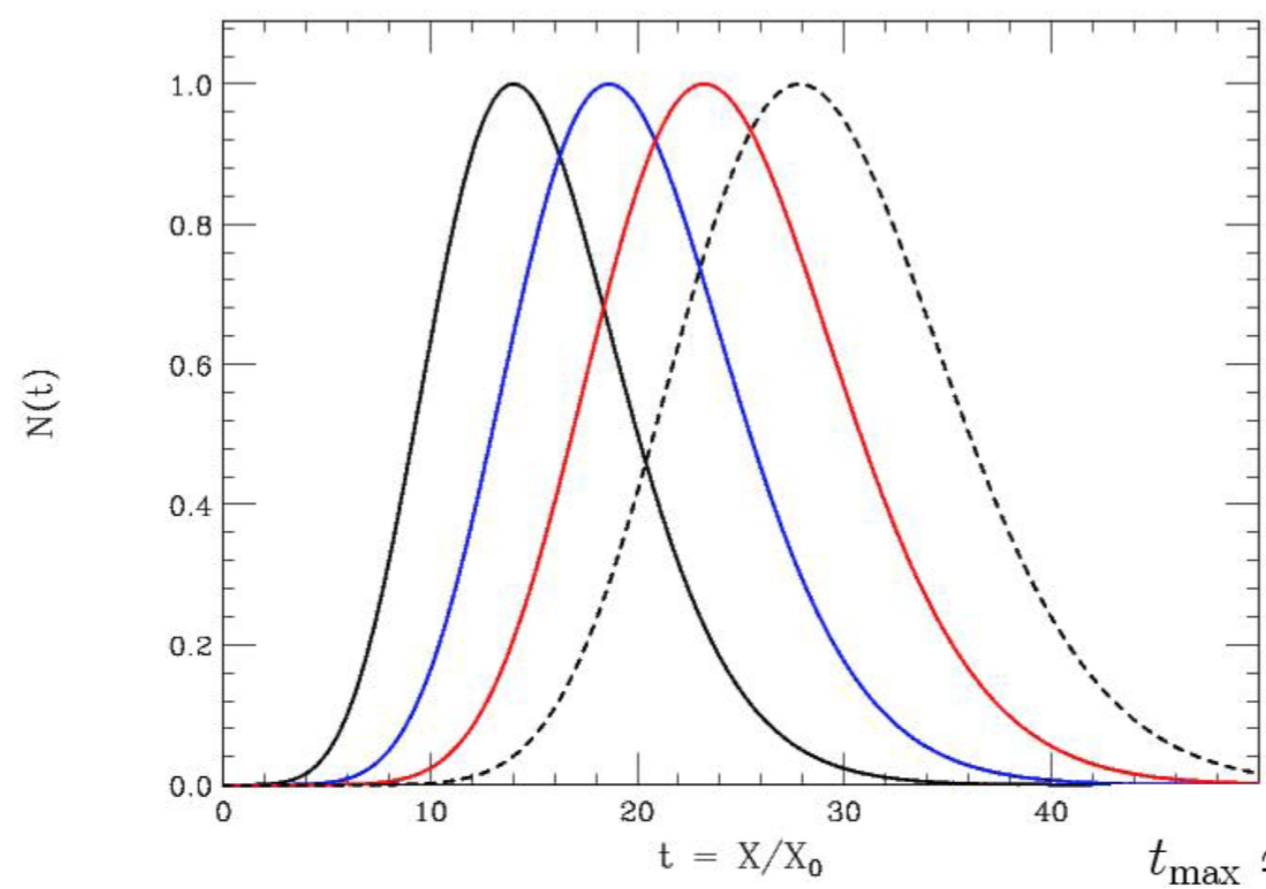
$$N(t) = N_{max}\,\exp\left[t\left(1 - \frac{3}{2}\ln\left(\frac{3t}{t+2t_{max}}\right)\right)\right]$$

electromagnetic showers

Greisen profile - **average** development of EM showers

$$N(t) = N_{max}\ \exp\left[t\left(1 - \frac{3}{2}\ln\left(\frac{3t}{t+2t_{max}}\right)\right)\right]$$

Greisen profile



$$t_{\max} \simeq \ln\left(\frac{E_0}{E}\right)$$

electromagnetic showers

Greisen profile - average development of EM showers

$$N_{\mathrm{Greisen}}(E_0, t) = \frac{0.135}{\sqrt{\ln(E_0/E_{min})}} \exp\left[t\left(1 - \frac{3}{2}\ln s\right)\right]$$

Approximation A

$$N_{\mathrm{Greisen}}(E_0, t) = \frac{0.31}{\sqrt{\ln(E_0/\epsilon)}} \exp\left[t\left(1 - \frac{3}{2}\ln s\right)\right]$$

Approximation B

- this approximate solution is valid for

$$\lambda_1(s) = \frac{1}{2}(s - 1 - 3\ln s)$$

- and for

$$t_{max} = \ln\left(\frac{E_0}{E_{\min}}\right) \qquad \epsilon < E_{\min} < E_0$$

- age

$$s = \frac{3t}{t + 2t_{max}}$$

estimate the **shape** of the particle spectrum knowing the **age** of the shower

electromagnetic showers

Gaisser-Hillas longitudinal profile

- empirical formula describing shower longitudinal profiles
- used to fit observations from fluorescence detection experiments

$$N_{\mathrm{GH}}(t) = N_{\max} \left(\frac{t - t_0}{t_{max} - t_0} \right)^{(t_{max} - t_0)/\Lambda} \exp\left[\frac{t_{max} - t_0}{\Lambda} \right]$$

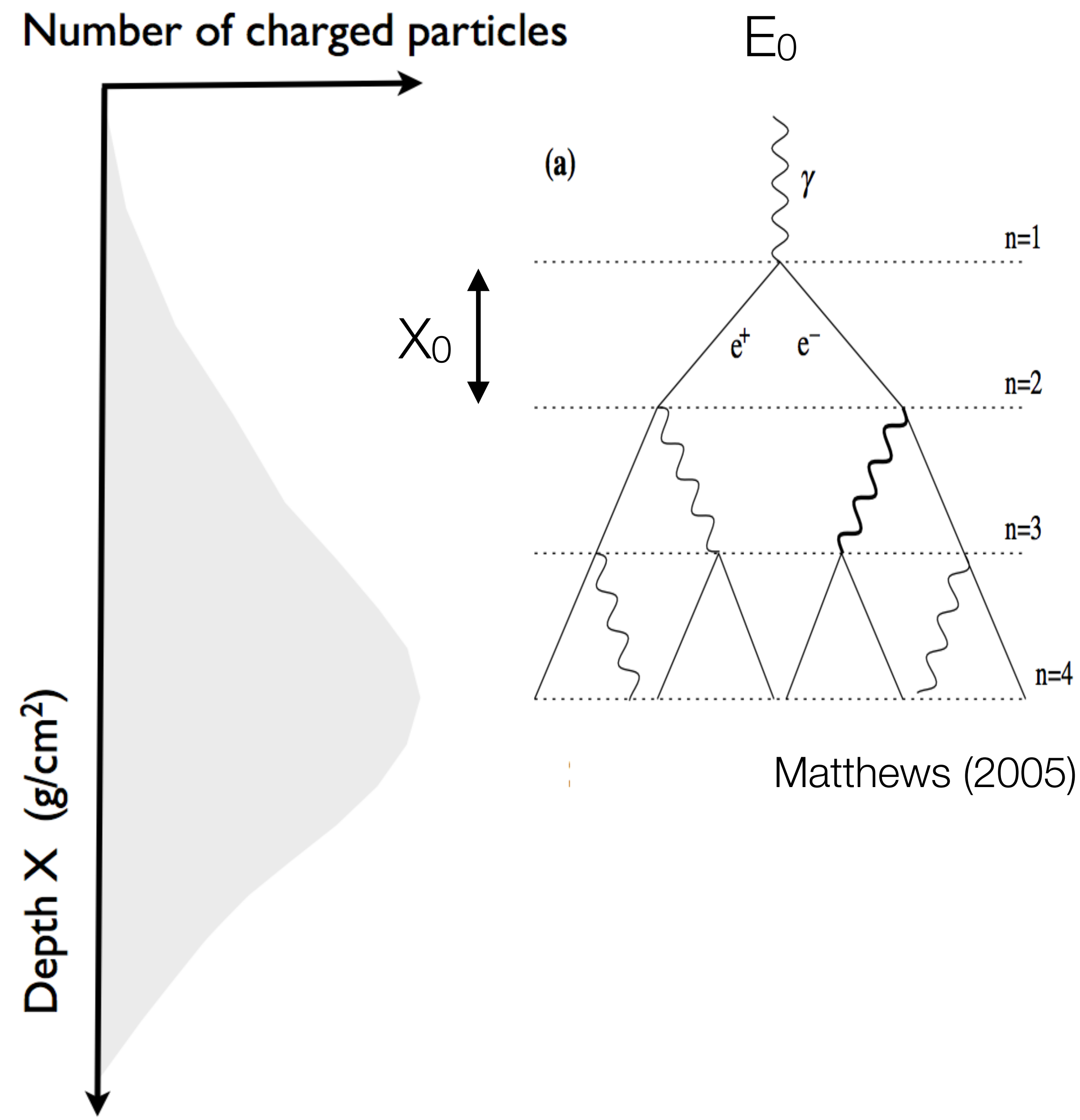
electromagnetic showers
Heitler model

- above MeV mostly pair production & bremsstrahlung
- radiation length

$$\frac{1}{\lambda_{rad}} = \frac{N_A}{A} 4Z(Z+1)\alpha r_e^2 \ln 183 Z^{-1/3}$$

- maximum shower depth @ $E_c = 2 m_e c^2$

$$X_{max} = X_0 \frac{\ln(E_0/E_c)}{\ln 2}$$



Matthews (2005)

Thank You

workshop in particle physics

hadronic showers

references

- **printed material**
  - *Cosmic Ray Theory* - Rossi & Greisen, Mod. Phys. 13, 240, 1941
  - *Cosmic Ray Showers* - Greisen, Annu. Rev. Nucl. Sci. 10, 63, 1960
  - *Concept of "age" and "universality" in cosmic ray showers* - Paolo Lipari, PRD 79, 063001, 2009
  - Cosmic Rays and Particle Physics. Thomas K. Gaisser