AN INTRODUCTION TO PLASMAPHYSICS

First MERCUR Winterschool on Plasma-Astroparticle Physics

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Contents

I. Defining A Plasma

1. Plasma parameter

1.1. Plasma?

An *overall neutral charged gas* containing so many free charged particles that their *collective Lorentz forces influence the properties* of the medium considerably.

1.2. Electron plasma frequency

The frequency of the electrons that oscillate around the average density due to the pulling Coulomb force and the inertia effect of the particles is called electron plasma frequency. Assuming the electrons (of density n_e) being shifted by δ with reference to the resting ions, the electric field yields $E = 4\pi\, e\, n_e \delta$. Using the equation of motion with respect to this electric field, we obtain (thermal motion is ignored!)

$$m_e \frac{d^2\delta}{dt^2} = -e\,E = -4\pi\, e^2\, n_e \delta. \tag{1}$$

Consequently, the eigenfrequency of this oscillation determines the *electron plasma frequency*

$$\omega_{\mathrm{pe}} \equiv \sqrt{\frac{4\pi\, e^2 n_e}{m_e}} = 5.64 \cdot 10^4 \sqrt{\frac{n_e}{1\,\mathrm{cm}^{-3}}}\,\mathrm{Hz}\,. \tag{2}$$

1.3. Debye length

The distance beyond which the Coulomb force of a single test ion is shielded by the surrounding plasma electrons is determined by the *Debye length*

$$\lambda_{\mathrm{De}} \equiv v_{th}/\omega_{\mathrm{pe}} = \sqrt{\frac{k_B T_e}{4\pi\, n_e e^2}} = 6.9 \sqrt{\frac{T_e/1\,\mathrm{K}}{n_e/1\,\mathrm{cm}^{-3}}}\,\mathrm{cm}\,. \tag{3}$$

Necessary requirements for a plasma

(i) In order to obtain an overall neutral charged gas the physical dimension L of the plasma must be much larger than the Debye length, i.e.

$$L \gg \lambda_{\mathrm{De}}\,. \tag{4}$$

(ii) In order to obtain a collective particle behavior the number of particles within the Debye sphere, which is defined by $\lambda_{\mathrm{s}} = \frac{4\pi}{3} n_e \lambda_{\mathrm{De}}^3$, has to be much more than one. The inversed number determines the so-called *plasma parameter* which subsequently yields the second criterion for the definition of a plasma

$$g \equiv \lambda_s^{-1} = 7.3 \cdot 10^{-4} \left(\frac{n_e}{1\,\mathrm{cm}^{-3}}\right)^{\frac{1}{2}} \left(\frac{T_e}{1\,\mathrm{K}}\right)^{-\frac{3}{2}} \ll 1\,. \tag{5}$$

Table I.1.: Estimate of cosmic plasma parameters (taken from Schlickeiser 2002)

System	Interst. gas	Mol. cloud	Sol. corona	AGN	Cluster of galaxies	Cosmic rays
n_e (cm^{-3})	0.1	1	10^6	10^9	10^{-2}	10^{-9}
T_e (K)	10^4	10^2	10^6	10^7	10^7	10^{12}
L (cm)	10^{18}	10^{17}	10^{10}	10^{15}	10^{23}	10^{18}
$\omega_{\mathrm{p}e}$ (s^{-1})	$2 \cdot 10^4$	$5 \cdot 10^4$	$5 \cdot 10^7$	$2 \cdot 10^9$	$5 \cdot 10^3$	2
$\lambda_{\mathrm{D}e}$ (cm)	$2 \cdot 10^3$	69	6.9	0.7	$2 \cdot 10^5$	$2 \cdot 10^{11}$
g	$3 \cdot 10^{-10}$	$7 \cdot 10^{-7}$	$7 \cdot 10^{-10}$	$7 \cdot 10^{-10}$	$2 \cdot 10^{-15}$	$2 \cdot 10^{-26}$

II. Kinetic Description Of Plasmas

1. Phase space density

Using the probability $F_j(\mathbf{x},\,\mathbf{v})$ of finding a particle j with the velocity $\mathbf{v}$ located at the point $\mathbf{x}$, so that $0 \leq F_j(\mathbf{x},\,\mathbf{v}) \leq 1$, the probability density function or *phase space density* $f_j(\mathbf{x},\,\mathbf{v})$ is obtained by

$$f_j(\mathbf{x},\,\mathbf{v}) = \frac{\partial^3 F_j}{\partial v_x\,\partial v_y\,\partial v_z}\,. \tag{1}$$

2. Vlasov equation

In the following, the phase space density f_j is considered to change in time due to the influence of the Lorentz force. Based on the conservation of f_j in phase space (no particle loss or generation in phase space) we become

$$\frac{\mathrm{d}f_j}{\mathrm{d}t} = \frac{\partial f_j}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f_j}{\partial \mathbf{x}} + \dot{\mathbf{v}} \cdot \frac{\partial f_j}{\partial \mathbf{v}} = 0\,, \tag{2}$$

where $\dot{\mathbf{x}} = \mathbf{v}$, $\partial/\partial\mathbf{x} = \nabla_x$ and $\partial/\partial\mathbf{v} = \nabla_v$. Using the Lorentz force

$$m_j\,\dot{\mathbf{v}}_j = q_j \left(\mathbf{E}(\mathbf{x},\,t) + \frac{\mathbf{v}_j \times \mathbf{B}(\mathbf{x},\,t)}{c}\right) \tag{3}$$

as well as a source term $S_j(\mathbf{x},\,\mathbf{v},\,t)$ representing additional sources and sinks of particles, Eq. (2) yields the *Vlasov equation* (or collisionless Boltzmann equation)

$$\frac{\mathrm{d}f_j}{\mathrm{d}t} = \frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \nabla_x f_j + \frac{q}{m}\left(\mathbf{E}(\mathbf{x},\,t) + \frac{\mathbf{v} \times \mathbf{B}(\mathbf{x},\,t)}{c}\right) \cdot \nabla_v f_j = S_j(\mathbf{x},\,\mathbf{v},\,t)\,. \tag{4}$$

It becomes useful to introduce some physical quantities. Therefore, the space-averaged number density n_a of particles of species a is used and we define

$$\begin{aligned}
&\text{(i) the number density of plasma particles } n_a(\mathbf{x},\,t) = n_a \int_{-\infty}^{\infty} \mathrm{d}^3v\, f_a(\mathbf{x},\,\mathbf{v},\,t)\,,\\
&\text{(ii) the flux of plasma particles } \mathbf{j}_a(\mathbf{x},\,t) = n_a \int_{-\infty}^{\infty} \mathrm{d}^3v\, \mathbf{v}\, f_a(\mathbf{x},\,\mathbf{v},\,t)\,,\\
&\text{(iii) the mean velocity of plasma particles } \mathbf{V}_a(\mathbf{x},\,t) \equiv \frac{\int_{-\infty}^{\infty} \mathrm{d}^3v\, \mathbf{v} f_a(\mathbf{x},\,\mathbf{v},\,t)}{n_a(\mathbf{x},\,t)}\,,\\
&\text{(iv) the pressure tensor } \Pi_{a,ik}(\mathbf{x},\,t) \equiv m_a \int_{-\infty}^{\infty} \mathrm{d}^3v\, f_a(\mathbf{x},\,\mathbf{v},\,t)(v_i - V_i)(v_k - V_k)\,.
\end{aligned} \tag{5}$$

Moments of the Vlasov equation

Taking the *0th velocity moment* of the Vlasov Eq. (4) yields

$$\underbrace{\int_{-\infty}^{\infty} \mathrm{d}^3 v \, \frac{\partial f_a}{\partial t}}_{=\frac{\partial}{\partial t} n_a(\mathbf{x},t)} + \underbrace{\int_{-\infty}^{\infty} \mathrm{d}^3 v \, \mathbf{v} \cdot \nabla_x f_a}_{=\nabla_x \cdot \mathbf{j}_a(\mathbf{x},t)} + \underbrace{\frac{q_a}{m_a} \int_{-\infty}^{\infty} \mathrm{d}^3 v \left(\mathbf{E}(\mathbf{x},\, t) + \frac{\mathbf{v} \times \mathbf{B}(\mathbf{x},\, t)}{c} \right) \cdot \nabla_v f_a}_{=0} = \underbrace{\int_{-\infty}^{\infty} \mathrm{d}^3 v \, S_a(\mathbf{x},\, \mathbf{v},\, t)}_{\equiv Q_a(\mathbf{x},t)} , \tag{6}$$

where the second term simplifies due to $\nabla_x \mathbf{v} = 0$ and the third term vanishes since in the case of the x component (and subsequently also the other components) f_a vanishes at $v_x = \pm\infty$ and $\mathbf{E}$, $\mathbf{B}$, v_y, v_z do not depend on v_x. Thus, the 0th moment of the Vlasov equation yields the *continuity equation*

$$\frac{\partial n_a(\mathbf{x},\, t)}{\partial t} + \nabla_x \cdot \mathbf{j}_a(\mathbf{x},\, t) = Q_a(\mathbf{x},\, t) \,. \tag{7}$$

The *1st velocity moment* of the Vlasov equation leads (shown in Schlickeiser 2002) to the *momentum equation*

$$m_a n_a(\mathbf{x},\, t) \left(\frac{\partial \mathbf{V}_a}{\partial t} + \mathbf{V}_a \nabla_x \mathbf{V}_a \right) + \sum_{i=1}^{3} \nabla_{x_i} \Pi_{a,ki} = n_a(\mathbf{x},\, t)\, q_a \left(\mathbf{E} + \frac{\mathbf{V}_a \times \mathbf{B}}{c} \right) + Q\mathbf{P}_a(\mathbf{x},\, t) \,, \tag{8}$$

where $Q\mathbf{P}_a(\mathbf{x},\, t) \equiv m_a \int_{-\infty}^{\infty} \mathrm{d}^3 v \, (\mathbf{v} - \mathbf{V}_a)\, S_a(\mathbf{x}, \mathbf{v},\, t)$ denotes the momentum source term.
Note that an infinite number of equations as derived from the Vlasov equation is needed in order to obtain a complete description of the plasma, since the nth moment of the Vlasov equation will always involve a term with $n+1$ factors of $\mathbf{v}$. In practice, this series of equations is truncated by physical arguments, like the cold plasma approximation where Π_a vanishes or the description of the flux vector via diffusion and advection approximations, as shown in the following chapter.

III. The Transport Equation

1. Convection-diffusion equation

The flux $\mathbf{j}_a(\mathbf{x}, t)$ is determined by
(i) *diffusion*, which is typically approximated by the Fick's first law (where the diffusive flux is proportional to the local concentration gradient)

$$\mathbf{j}_{diff}(\mathbf{x}, t) = -\kappa \, \nabla_x n_a(\mathbf{x}, t) \,, \tag{1}$$

with the diffusion coefficient κ as well as
(ii) *advection*, which describes the bulk flow due to a velocity field $\mathbf{u}$, so that we obtain

$$\mathbf{j}_{adv}(\mathbf{x}, t) = n_a(\mathbf{x}, t) \, \mathbf{u}(\mathbf{x}, t) \,. \tag{2}$$

Using Eq.(1) and (2) the continuity equation (7) leads to the *convection-diffusion equation* (convection means either advection or diffusion or a combination of both of them)

$$\frac{\partial n_a(\mathbf{x}, t)}{\partial t} = Q_a(\mathbf{x}, t) + \nabla_x \left(\kappa \, \nabla_x n_a(\mathbf{x}, t)\right) - \nabla_x \left(n_a(\mathbf{x}, t) \, \mathbf{u}(\mathbf{x}, t)\right) \,. \tag{3}$$

2. Inclusion of momentum losses

The convection-diffusion equation (3) describes the spatial propagation of particles with time but there are no momentum loss processes considered so far. Thus, interactions (e.g. bremsstrahlung, synchrotron radiation, inverse Compton interactions or pion production) with ambient target photon and matter fields have to be taken into account when these particle interaction processes operate on time scales of the same order of magnitude as the diffusion and advection time scales. The loss processes have to be divided into two groups:
(i) Continuous loss processes (where the particle loses momentum but the total number of particles is conserved) are taken into account by the additional term

$$-\frac{\partial}{\partial p} \left(\dot{p} \, n_a\right) , \tag{4}$$

with the momentum p of the particles and the momentum loss rate $\dot{p}$.
(ii) Catastrophic loss processes (where the total number of particles changes) are included by the additional term

$$-\frac{n_a}{\tau} \,, \tag{5}$$

where τ denotes the appropriate loss time.
In total, the convection diffusion equation under consideration of momentum losses, which determines the temporal development of the number density of plasma particles $n_a(\mathbf{x}, p, t)$ yields

$$\boxed{\frac{\partial n_a}{\partial t} = Q_a(\mathbf{x}, t) + \nabla_x \left(D \, \nabla_x n_a\right) - \nabla_x \left(n_a \, \mathbf{u}(\mathbf{x}, t)\right) - \frac{\partial}{\partial p} \left(\dot{p} \, n_a\right) - \frac{n_a}{\tau} \,.} \tag{6}$$

3. Leaky box model

Here we assume that the particles (of number density n) propagate freely in a *containment volume* with a *(constant) time of escape* τ_{esc}. Thus, diffusion and advection terms vanish, as the time scales τ_{diff}, τ_{adv} of the effective diffusion and advection, respectively, are considered to be much bigger than τ_{esc} and spatially averaged quantities can be considered. The Eq. (6) is simplified under these assumptions to

$$\frac{\partial n}{\partial t} = Q - \frac{n}{\tau_{esc}} \tag{7}$$

when there is also no momentum changing process ($\dot{p} = 0$). Using a delta function source $Q(p, t) = n_0(p)\,\delta(t)$ the differential equation (7) yields the simple solution

$$n(p, t) = n_0(p)\,\exp(-t/\tau_{esc})\,. \tag{8}$$

Using a steady state approximation, where $\partial n/\partial t \simeq 0$ the particle number density yields

$$n(E) = \tau_{esc}(E)\,Q(E)\,. \tag{9}$$

Consequently, the initial energy spectrum can be changed by propagation effects. In the simple case that the escape time is approximated by the Larmor radius R_L according to $\tau_{esc} \simeq R_L/c = E/(q\,B\,c^2)$ an initial power law spectrum $Q(E) = Q_0\,E^{-\alpha}$ with $\alpha < 3$ is flattened to $n(E) \propto E^{-\alpha+1}$. However the cosmic ray spectrum Fig. (III.1) shows that below the knee $1 - \alpha \simeq -2.67$, so that other effects (like diffusion) need to be included in order to describe the observed spectrum.

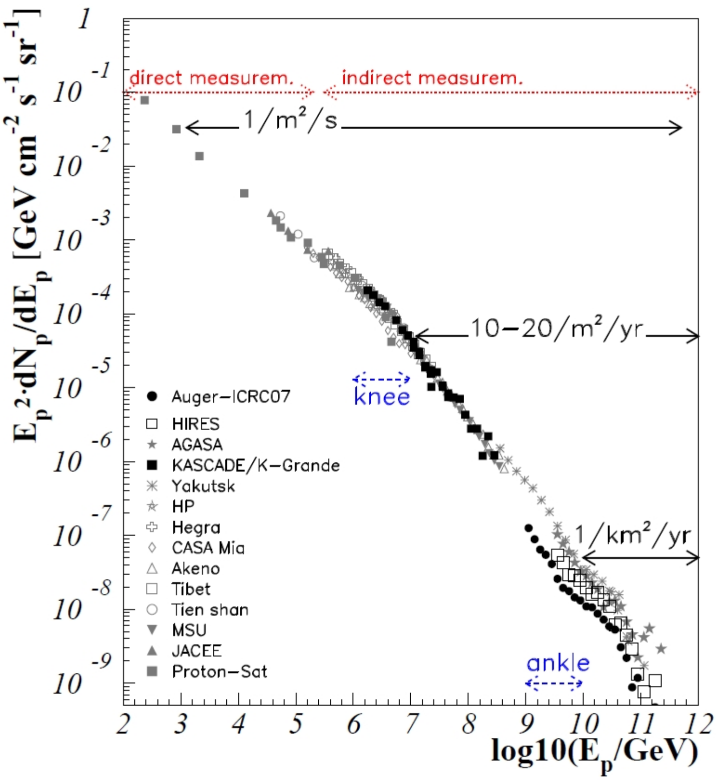

Figure III.1.: The weighted cosmic ray spectrum, where the kind of observation, the kinks in the spectral slope, as well as the approximated particle flux per time and area are indicated (Becker 2008).

IV. Quasilinear Theory & Parallel Scattering

The Vlasov equation (as derived in section II.2) leads to the fundamental problem of plasma physics: The plasma particles determine the electromagnetic fields (according to the four Maxwell equations) and vice versa (according to the Lorentz force). In order to proceed with a solution of this coupled problem two opposite points of view need to be taken (Schlickeiser 2002):

1. The *test wave approach*, in which the plasma particle distribution functions are assumed to be given in a prescribed initial state, so that the resulting electromagnetic field and their property can be discussed. This approach leads to different types of plasma waves characterized by the dispersion relation, that depends on the initial plasma condition. We are not going to focus on this topic, so please have a look at Schlickeiser 2002 for more details.

2. The *test particle approach* , in which the electromagnetic fields are assumed to be given, and the response of the particles can be discussed. This approach is studied in the following.

1. Fokker-Planck equation

Subsequently, we start with the Vlasov equation and take the effects of the plasma waves on the particles of sort a into account

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\,\frac{\partial f_a}{\partial \mathbf{x}} + \dot{\mathbf{p}}\,\frac{\partial f_a}{\partial \mathbf{p}} = S_a(\mathbf{x},\,\mathbf{p},\,t)\,. \tag{1}$$

with $\dot{\mathbf{p}} = q_a\left(\mathbf{E}_T(\mathbf{x},\,t) + \frac{\mathbf{v}\times\mathbf{B}_T(\mathbf{x},t)}{c}\right)$ and $\dot{\mathbf{x}} = \mathbf{v} = \mathbf{p}/(\gamma\, m_a)$. Under consideration of plasma turbulence ($\delta\mathbf{E}\,,\;\delta\mathbf{B}$) the total electromagnetic field yields

$$\mathbf{B}_T = \mathbf{B}_0 + \delta\mathbf{B}(\mathbf{x},\,t) \;\text{ and }\; \mathbf{E}_T = \delta\mathbf{E}\,, \tag{2}$$

where large-scale electric fields are negligible due to the high conductivity of cosmic plasmas and the uniform magnetic field is determined by $\mathbf{B}_0 = B_0\,\mathbf{e}_z$ with $B_0 \gg \delta\mathbf{B}$. Since the actual position due to the gyration of the particle in the uniform magnetic field is not so much of an interest as the coordinates of the guiding center

$$\mathbf{R} = (X,\,Y,\,Z) = \mathbf{x} + \frac{\mathbf{v}\times\mathbf{e}_z}{\epsilon\,\Omega}\,, \tag{3}$$

with the particle's gyrofrequency $\Omega = e\,B_0/(\gamma\,m_a\,c)$ and the charge sign $\epsilon = q_a/\,|q_a|$. Using spherical coordinates $(p,\,\mu,\,\phi)$ in momentum space

$$p_x = p\,\cos\phi\,\sqrt{1-\mu^2}\,,\quad p_y = p\,\sin\phi\,\sqrt{1-\mu^2}\,,\quad p_z = p\,\mu\,, \tag{4}$$

so that the spatial coordinates (3) of the guiding center become

$$X = x + \frac{v\sqrt{1-\mu^2}}{\epsilon\Omega}\,\sin\phi\,,\quad Y = y - \frac{v\sqrt{1-\mu^2}}{\epsilon\Omega}\,\cos\phi\,,\quad Z = z\,. \tag{5}$$

Applying the new coordinate set $x_\sigma = (p,\, \mu,\, \phi,\, X,\, Y,\, Z)$ to the Vlasov equation (1) it yields (with the Einstein summation convention)

$$\frac{\partial f_a}{\partial t} + v\,\mu\,\frac{\partial f_a}{\partial Z} - \epsilon\,\Omega\,\frac{\partial f_a}{\partial \phi} + \frac{1}{p^2}\,\frac{\partial}{\partial x_\sigma}\,\left(p^2 g_{x_\sigma} f_a\right) = S_a(\mathbf{x},\, \mathbf{p},\, t)\,, \tag{6}$$

where the generalized force term $g_{x_\sigma} = (\dot{p},\, \dot{\mu},\, \dot{\phi},\, \dot{X},\, \dot{Y},\, \dot{Z})$ includes the effects of the randomly fluctuating electromagnetic fields. In the following, an ensemble of distribution functions is considered in order to find an *expectation value* of f_a, i.e. $\langle f_a(\mathbf{x},\, \mathbf{p},\, t)\rangle = F_a(\mathbf{x},\, \mathbf{p},\, t)$. Using $\langle \delta\mathbf{B}(\mathbf{x},\, t)\rangle = \langle \delta\mathbf{E}(\mathbf{x},\, t)\rangle = 0$ (so that $\langle \mathbf{B}(\mathbf{x},\, t)\rangle = B_0$ and $\langle \mathbf{E}(\mathbf{x},\, t)\rangle = 0$) we obtain

$$\frac{\partial F_a}{\partial t} + v\,\mu\,\frac{\partial F_a}{\partial Z} - \epsilon\,\Omega\,\frac{\partial F_a}{\partial \phi} = S_a(\mathbf{x},\, \mathbf{p},\, t) - \frac{1}{p^2}\,\frac{\partial}{\partial x_\sigma}\,\left(\langle p^2 g_{x_\sigma}\,\delta f_a\rangle\right)\,, \tag{7}$$

where $\delta f_a(\mathbf{x},\, \mathbf{p},\, t) = f_a(\mathbf{x},\, \mathbf{p},\, t) - F_a(\mathbf{x},\, \mathbf{p},\, t)$ denotes the fluctuation of the distribution function, which is determined by

$$\frac{\partial \delta f_a}{\partial t} + v\,\mu\,\frac{\partial \delta f_a}{\partial Z} - \epsilon\,\Omega\,\frac{\partial \delta f_a}{\partial \phi} = -g_{x_\sigma}\,\frac{\partial F_a}{\partial x_\sigma} - g_{x_\sigma}\,\frac{\partial \delta f_a}{\partial x_\sigma} + \langle g_{x_\sigma}\,\frac{\partial \delta f_a}{\partial x_\sigma}\rangle\,. \tag{8}$$

It can be shown (Schlickeiser 2002) that the variation δf_a generated by g_{x_σ} must remain much smaller than F_a, so that the first term on the right hand side of Eq. (8) is dominating and the remaining differential equation can be solved by the method of characteristics, which yields

$$\delta f_a(t) = \delta f_a(t_0) - \int_{t_0}^{t} \mathrm{d}s\,\left[g_{x_\sigma}(x_\sigma,\, s)\,\frac{\delta F_a(x_\sigma,\, s)}{\partial x_\sigma}\right]'\,. \tag{9}$$

The prime indicates that the bracketed term has to be evaluated along the characteristics, i.e. an unperturbed particle orbit in the uniform magnetic field. The particle's phase space density at the initial time t_0 is considered as completely uncorrelated to the turbulent field so that $\langle \delta f_a g_{x_\sigma}\rangle = 0$ and after some further rearrangements the Vlasov equation (7) yields the *Fokker-Planck equation*

$$\boxed{\frac{\partial F_a}{\partial t} + v\,\mu\,\frac{\partial F_a}{\partial Z} - \epsilon\,\Omega\,\frac{\partial F_a}{\partial \phi} = S_a(\mathbf{x},\, \mathbf{p},\, t) + \frac{1}{p^2}\,\frac{\partial}{\partial x_\sigma}\,\left(p^2 D_{x_\sigma x_\nu}\,\frac{\partial F_a}{\partial x_\nu}\right)\,,} \tag{10}$$

with the 25 *Fokker-Planck coefficients*

$$D_{x_\sigma x_\nu}(x_\eta,\, t) = \int_0^t \mathrm{d}s\,\langle \bar{g}_{x_\sigma}(t)\,\bar{g}_{x_\nu}(s)\rangle\,. \tag{11}$$

These are homogeneous integrals along the unpertubed particle orbits of the fluctuating force field. In the next section, we focus on the scattering of cosmic rays parallel to the background magnetic field which results from pitch-angle diffusion in phase space.

2. Parallel transport of cosmic rays

Starting with the two-dimensional Fokker-Planck equation

$$\frac{\partial F}{\partial t} + v\,\mu\,\frac{\partial F}{\partial z} = \frac{\partial}{\partial \mu}\,\left(D_{\mu\mu}\,\frac{\partial F}{\partial \mu}\right)\,, \tag{12}$$

with the pitch-angle Fokker-Planck coefficient

$$D_{\mu\mu} = \int_0^\infty \mathrm{d}t\,\langle \dot{\mu}(t)\,\dot{\mu}(0)\rangle\,. \tag{13}$$

These formulas can be deduced from the more general ones above in case of negligible perpendicular Fokker-Planck coefficients as well as momentum diffusion due to the assumption of purely magnetic fluctuations (Shalchi 2009).
Applying the operator $\frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu$ onto the Fokker-Planck equation (12) yields the continuity equation

$$\frac{\partial M(z,t)}{\partial t} = -\frac{\partial j(z,t)}{\partial z} \tag{14}$$

where we defined the pitch angle averaged particle density

$$M(z,t) \equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu\, F(\mu,z,t) \tag{15}$$

as well as the pitch angle averaged current density

$$j(z,t) = \frac{v}{2}\int_{-1}^{1}\mathrm{d}\mu\, \mu F(\mu,z,t) \tag{16}$$

and used $D_{\mu\mu}(\mu=\pm 1)=0$ since $g_\mu(\mu=\pm 1)=0$ (see Exercise 2.1). With partial integration the current density (16) can be rewritten as

$$j(z,t) = -\frac{v}{4}\int_{-1}^{1}\mathrm{d}\mu\, \frac{\partial(1-\mu^2)}{\partial\mu}\, F(\mu,z,t) = \frac{v}{4}\int_{-1}^{1}\mathrm{d}\mu\,(1-\mu^2)\,\frac{F(\mu,z,t)}{\partial\mu}\,. \tag{17}$$

In order to replace the term $\frac{\partial F}{\partial\mu}$, the operator $\int_{-1}^{\mu}$ is applied onto the Fokker-Planck equation (12) and the subsequent multiplication with $(1-\mu^2)/D_{\mu\mu}$ yields

$$\frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial}{\partial t}\int_{-1}^{\mu}\mathrm{d}\nu\, F(\nu,z,t) + \frac{1-\mu^2}{D_{\mu\mu}}v\frac{\partial}{\partial z}\int_{-1}^{\mu}\mathrm{d}\nu\,\nu F(\nu,z,t) = (1-\mu^2)\,\frac{\partial F(\mu,z,t)}{\partial\mu}\,. \tag{18}$$

Therewith, the current density (17) is expressed by

$$j(z,t) = \frac{v}{4}\int_{-1}^{1}\mathrm{d}\mu\,\frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu}\mathrm{d}\nu\,\frac{\partial F(\nu,z,t)}{\partial t} + \frac{v^2}{4}\int_{-1}^{1}\mathrm{d}\mu\,\frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu}\mathrm{d}\nu\,\nu\frac{\partial F(\nu,z,t)}{\partial z}\,. \tag{19}$$

At late times $t\to\infty$ the ensemble-averaged particle distribution function $F(\mu,z,t)$ is equal to the pitch angle averaged particle density $M(z,t)$ due to pitch angle isotropization and thus

$$\begin{aligned} j(z,t) &= \frac{v}{4}\frac{\partial M(z,t)}{\partial t}\int_{-1}^{1}\mathrm{d}\mu\,\frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu}\mathrm{d}\nu + \frac{v^2}{4}\frac{\partial M(z,t)}{\partial z}\int_{-1}^{1}\mathrm{d}\mu\,\frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu}\mathrm{d}\nu\,\nu \\ &= \kappa_{zt}\,\frac{\partial M(z,t)}{\partial t} - \kappa_{zz}\,\frac{\partial M(z,t)}{\partial z}\,, \end{aligned} \tag{20}$$

with

$$\kappa_{zt} = \frac{v}{4}\int_{-1}^{1}\mathrm{d}\mu\,\frac{(1-\mu^2)(1+\mu)}{D_{\mu\mu}} \quad \text{and } \kappa_{zz} = \frac{v^2}{8}\int_{-1}^{1}\mathrm{d}\mu\,\frac{(1-\mu^2)^2}{D_{\mu\mu}}\,. \tag{21}$$

Using the continuity equation (14) we obtain

$$j(z,t) = -\frac{\partial}{\partial z}\left[\kappa_{zt}j(z,t) + \kappa_{zz}M(z,t)\right]\,. \tag{22}$$

Comparing Eq. (15) with (16) it can be seen that $j(z,t)$ tends to zero for late times since $f(\mu,z,t)$ becomes almost isotropic and the current density (22) simplifies to

$$j(z,t) = -\kappa_{zz}\,\frac{\partial M(z,t)}{\partial z} \tag{23}$$

and the *diffusion equation* yields

$$\boxed{\frac{\partial M(z,t)}{\partial t} = \kappa_{zz} \frac{\partial^2 M(z,t)}{\partial z^2}} \tag{24}$$

in the case of a diffusion coefficient κ_{zz} independent of z. Thus, the resulting equation is equal (apart from the source and advection term) to the convection-diffusion equation we already derived in section III.1, however, without using Fick's law and under consideration of the effect of magnetic turbulences on the cosmic rays.

V. Calculation Of The Diffusion Coefficient

In this chapter the diffusion coefficient κ_{zz} is going to be calculated in detail, however, there first need to be some assumptions about the electromagnetic fluctuation field.

1. Alfvén waves

An *Alfvén wave* is a type of magnetohydrodynamic wave in which (not only electrons but also) ions oscillate (but with a different velocity) in response to a restoring force. Thereby, the oscillation travels with a low frequency (compared to the gyrofrequency of the ions), i.e. the wavenumber $|k| \ll k_c \equiv 8.78 \cdot 10^{-8} \sqrt{n_e/(1\,\mathrm{cm}^{-3})}\,\mathrm{cm}^{-1}$, along the ordered magnetic field (although there are also waves propagating perpendicular to the magnetic field, which are called *magnetosonic waves*). The wave is dispersionless

$$\omega_j = jV_A\,k_\parallel\,, \quad \text{with } j = \pm 1 \tag{1}$$

where the *Alfvén velocity* is defined by

$$V_A \equiv \frac{B_0}{\sqrt{4\pi\,(m_p + m_e)\,n_e}} = 2.18 \cdot \left(\frac{B_0}{1\,\mathrm{G}}\right)\left(\frac{n_e}{1\,\mathrm{cm}^{-3}}\right)^{-1/2} \mathrm{cm/s}\,. \tag{2}$$

In the case of large wavenumbers $|k| \gg k_c$ the left-handed Alfvén waves become left-handed *ion cyclotron waves* with a k-independent frequency $\omega_j \simeq \Omega_p$ that only depends on the proton gyrofrequency. Additionally, the right-handed Alfvén waves develope at frequencies $\Omega_p < |\omega| < \Omega_e$ into the right-handed *whistler waves*, which are dispersive since $\omega_j \simeq -(2V_A k^2/k_c + \Omega_p)$. Using only these parallel propagating magnetohydrodynamic waves it can be shown (Achatz et al. 1991) that the pitch-angle Fokker-Planck coefficient simplifies to

$$D_{\mu\mu} = \frac{\Omega^2(1-\mu^2)}{2B_0^2} \sum_{j=\pm1} \sum_{n=-\infty}^{\infty} \int \mathrm{d}^3k\, R_j(\mathbf{k}, \omega_j) \left[1 - \frac{\mu\omega_j}{k_\parallel v}\right]^2 \left(J_{n+1}^2(W)\, P_{RR}^j(\mathbf{k}) + J_{n-1}^2(W)\, P_{LL}^j(\mathbf{k})\right)\,, \tag{3}$$

where the Breit-Wigner type resonance function $R_j(\mathbf{k}, \omega_j)$ is in the considered case of *undamped waves* determined by

$$R_j(\mathbf{k}, \omega_j) \equiv \pi\,(v\mu k_\parallel - \omega_j + n\Omega)\,. \tag{4}$$

Furthermore, $J_n(W)$ denotes the Bessel function of first kind with

$$W \equiv \frac{kv\sqrt{1-\mu^2}\,\sin\theta}{\Omega} \tag{5}$$

and the magnetic fluctuation correlation tensor

$$P_{lm}^j(\mathbf{k}) = \langle \delta\mathbf{B}_l(\mathbf{k})\,\delta\mathbf{B}_m(\mathbf{k})\rangle \tag{6}$$

is used for left-(L) and right-(R) handed polarized fluctuating magnetic field $\delta\mathbf{B}_{L,R} \equiv (\delta\mathbf{B}_x \pm \mathrm{i}\,\delta\mathbf{B}_y)/\sqrt{2}$. In order to calculate the Fokker-Planck coefficient (3) the turbulence geometry needs to be specified.

1.1. Slab turbulence

Here, we assume that the turbulent fields depend only on the z-coordinate which is parallel to the background magnetic field $\mathbf{B}_0 = B_0\,\mathbf{e}_z$, so that the *slab turbulence tensor* yields

$$P^j_{lm}(\mathbf{k}) = \begin{cases} g^j_s(k_\parallel)(\delta(k_\perp)/k_\perp)\,[\delta_{lm} + \mathrm{i}\,\sigma^j(k_\parallel)\,\epsilon_{lmz}]\,, & \text{for } l,m = x,y \\ 0\,, & \text{for } l,m = z\,, \end{cases} \tag{7}$$

where the Kronecker δ-symbol δ_{lm} as well as the total antisymmetric tensor ϵijl of rank 3 is used and $\sigma^j(k_\parallel)$ denotes the magnetic helicity ($\sigma = 0$ for linear polarized waves). The wave intensity function $g^j_s(k_\parallel)$ is determined by

$$(\delta B_j)^2 = \int \mathrm{d}^3k\,\left(P^j_{11}(\mathbf{k}) + P^j_{22}(\mathbf{k}) + P^j_{33}(\mathbf{k})\right) = 4\pi \int_{-\infty}^{\infty} \mathrm{d}k_\parallel\, g^j_s(k_\parallel)\,. \tag{8}$$

Considering only the Alfvénic part of the dispersion relation the Fokker-Planck coefficient (3) further reduces to

$$D_{\mu\mu} = \sum_{j=\pm 1} \frac{\pi^2\Omega^2(1-\mu^2)}{v\,B_0^2}\,\frac{(1-j\eta\mu)^2}{|\mu - j\eta|}\,\left[(1-\sigma^j(k^j_r))g^j_s(k^j_r) + (1+\sigma^j(-k^j_r))g^j_s(-k^j_r)\right]\,, \tag{9}$$

with the resonant parallel wavenumber $k^j_r = \frac{\Omega/v}{\mu - j\eta} = \frac{R_L^{-1}}{\mu - j\eta}$, where $\eta = V_A/v$ and R_L denotes the Larmor radius.

1.2. Kolmogorov-type power law turbulence spectra

In order to illustrate the pitch-angle diffusion coefficient κ_{zz}, the turbulence spectrum g^j_s as well as the helicity σ^j in the Fokker-Planck coefficient (9) still need to be specified. Due to Fig. (V.1) it is justifiable to assume (i) *power law turbulence spectra*

$$g^j_s(k_\parallel) = g^j_{s0}\,k_\parallel^{-q}\,, \quad \text{for } k_\parallel > k_{\parallel,\mathrm{min}} \quad \text{and} \quad g^j_i(k) = g^j_{i0}\,k^{-q}\,, \quad \text{for } k > k_{\mathrm{min}}\,, \tag{10}$$

with $q > 1$ (in the case of Kolmogorov turbulence $q = 5/3$) and

$$g^j_{s0} = \frac{q-1}{4\pi}\,(\delta B_j)^2\,k^{q-1}_{\parallel\mathrm{min}}\,, \quad g^j_{s0} = (q-1)\,(\delta B_j)^2\,k^{q-1}_{\mathrm{min}} \tag{11}$$

as well as (ii) *isospectral turbulence*, so that the helicities (i.e. σ^j, the normalized cross helicity state $H_c = 2g^{j=1}/g_{tot} - 1$ which denotes the ratio of the intensities of forward (j=1 or f) and backward (j=-1 or b) propagating waves and the fractional abundance of forward moving waves $r = (1 + H_c)/2$) are independent of wavenumber k.
Consequently, κ_{zz} can be determined and with $\lambda = 3\kappa_{zz}/v$ the cosmic-ray mean free path along the ordered magnetic field yields

$$\boxed{\lambda = \frac{3v}{8}\int_{-1}^{1}\mathrm{d}\mu\,\frac{(1-\mu^2)^2}{D_{\mu\mu}} = \frac{3}{2\pi(q-1)}\left(\frac{B_0}{\delta B}\right)^2 (R_L k_{\mathrm{min}})^{1-q} R_L\,F(\epsilon, H_c, \sigma_{f,b})} \tag{12}$$

with the integral

$$F(\epsilon, H_c, \sigma_{f,b}) \equiv \int_{-1}^{1}\mathrm{d}\mu\,\frac{1-\mu^2}{N(\mu)} \tag{13}$$

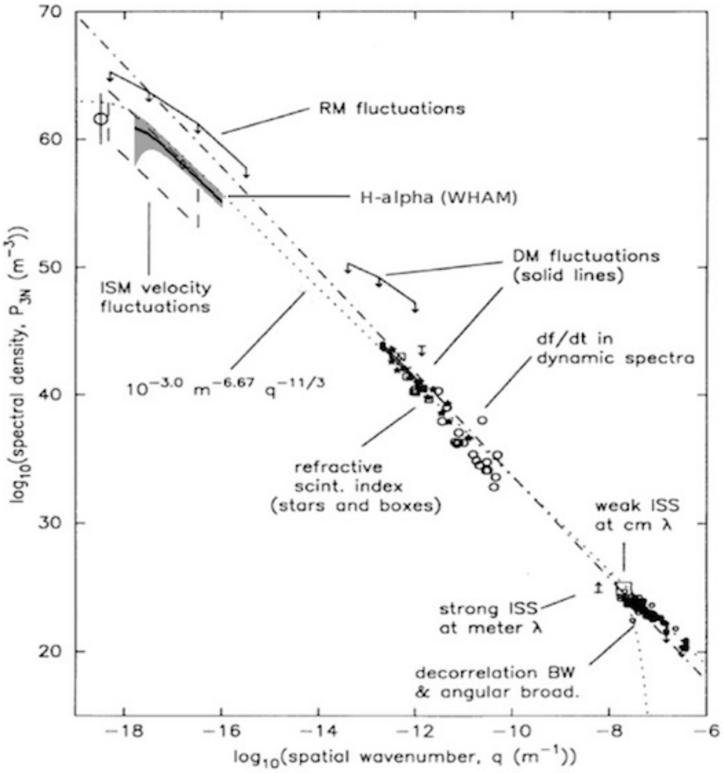

Figure V.1.: Different measurements indicate the power law structure of the interstellar turbulence.

where

$$\begin{aligned} N(\mu) =& r(1-\mu\eta)^2 \left|\mu-\eta\right|^{q-1} \left[(1+\sigma_f)\, H[\epsilon(\eta-\mu)] + (1-\sigma_f)\, H[\epsilon(\mu-\eta)]\right] \\ &+ (1-r)(1+\mu\eta)^2 \left|\mu+\eta\right|^{q-1} \left[(1+\sigma_b)\, H[\epsilon(-\eta-\mu)] + (1-\sigma_b)\, H[\epsilon(\mu+\eta)]\right] \end{aligned} \tag{14}$$

and still $\epsilon = q/\left|q\right| = \pm 1$. In the range $1 < q < 3$, it can be shown (Dung & Schlickeiser 1990) that for energetic particles $\eta = V_A/v \ll 1$, the integral is accurately approximated by

$$F(\epsilon, H_c, \sigma_{f,b}) \simeq \frac{2}{(2-q)(4-q)} \left[\frac{1}{a} + \frac{1}{\alpha}\right] + \eta^{2-q} \left[\frac{2}{d} - \frac{b}{a} - \frac{\beta}{\alpha} + \frac{1}{q-2}\left(\frac{1}{a} + \frac{1}{\alpha}\right)\right], \tag{15}$$

with

$$\begin{aligned} a =& r(1-\epsilon\sigma_f) + (1-r)(1-\epsilon\sigma_b)\,, \\ ab =& (1-r)(1-\epsilon\sigma_b) - r(1-\epsilon\sigma_f)\,, \\ \alpha =& r(1+\epsilon\sigma_f) + (1-r)(1+\epsilon\sigma_b)\,, \\ \alpha\beta =& r(1+\epsilon\sigma_f) - (1-r)(1+\epsilon\sigma_b)\,, \\ d =& r(1+\epsilon\sigma_f) + (1-r)(1-\epsilon\sigma_b)\,. \end{aligned} \tag{16}$$

In the case of a *linearly polarized* ($\sigma_f = \sigma_b = 0$) Alfvén wave field with *zero cross helicity* ($H_c = 0$, $r = 1/2$) the expression (15) further simplifies to

$$F \simeq \frac{4}{(2-q)(4-q)} + \eta^{2-q}\left(2 + \frac{2}{q-2}\right), \tag{17}$$

so that the mean free path (12) depends on the kinetic energy E_{kin} of the cosmic rays as follows:

1. At non-relativistic energies $\lambda(1 < q < 2) \propto E_{kin}^{(2-q)/2}$ and $\lambda(2 < q < 3)$ is constant.

2. At relativistic energies $\lambda(1 < q < 3) \propto E_{kin}^{2-q}$.

For $q < 2$ the mean free path is (apart from all helicity values) mainly determined by

$$\lambda \simeq R_L \left(\frac{B_0}{\delta B}\right)^2 \left(\frac{l_{\mathrm{m}ax}}{R_L}\right)^{q-1} , \tag{18}$$

where $l_{\mathrm{m}ax} = 2\pi\, k_{\mathrm{m}in}^{-1}$ denotes the maximal Alfvén wavelength. Since $l_{\mathrm{m}ax}$ has to be smaller than the size L of the astrophysical system we obtain

$$\boxed{\lambda \simeq R_L \left(\frac{B_0}{\delta B}\right)^2 \left(\frac{L}{R_L}\right)^{q-1} .} \tag{19}$$

Thus, in the case of *Kolmogorov diffusion* (where $q = 5/3$) the mean free path yields

$$\lambda_{Kol} \propto E_{kin}^{1/3} . \tag{20}$$

Going back to the simple steady state leaky box model of section III.3 and assuming that the particle escape is dominated by Kolmogorov diffusion, the escape time enlarges due to the effect of diffusion according to $\tau_{esc} \propto \lambda_{Kol}^{-1}$. Consequently, the observed cosmic ray spectrum yields $n(E) \propto E^{-\alpha-1/3}$ which gives a quite good agreement with the observations (see Fig. (III.1)). There is an elementary lower limit on the pitch angle diffusion given, by equalizing the mean free path with the Lamor radius of the cosmic rays, which is called *Bohm diffusion* (here $q = 1$).

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The references are appearing in the alphabetic order of the abbreviations and thus not necessarily in the alphabetic order of the name of the first author.

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